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Progenitors, Symmetric Presentations, and Related Topics

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Progenitors, Symmetric Presentations, and Related Topics

A Thesis

Presented to the

Faculty of

California State University,

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

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Abstract

A progenitor developed by Robert T. Curtis is a type of infinite groups formed by the semi-direct product of a free group m^{*n} and a transitive permutation group of degree n. To produce finite homomorphic images we had to add relations to the progenitor of the form $2^{*n} : N$. In this thesis we have investigated several permutations progenitors and monomials, $2^{*12} : S_4$, $2^{*12} : S_4 \times 2$, $2^{*13} : (13 : 4)$, $2^{*30} : ((2^{\bullet} : 3) : 5)$, $2^{*13} : 13$, $2^{*13} : (13 : 2)$, $2^{*13} : (13 : S_3)$, $53^{*2} :_m (13 : 4)$, $7^{*8} :_m (3^2 : 8)$, and $53^{*4} :_m (13 : 4)$. We have discovered that the permutations progenitors produced the following finite homomorphic images, we have found $PGL(2, 13)$, $U_3(4) : 2$, $2^{\bullet}S_z(8)$, $PSL(2, 7)$, $PGL(2, 27)$, $PSL(2, 8)$, $PSL(3, 3)$, $4^{\bullet}S_4(5)$, $PSL_2(53)$, and $13 : PGL_2(53)$ as homomorphic images of this progenitors. We will construct double coset enumeration for the homomorphic images, $2^{\bullet}S_z(8)$ over $(13 : 4)$ Suzuki twisted group, $PGL(2, 13)$ over S_4,and $PSL(2, 7)$ over S_4 and Maximal subgroups of $2 \times PGL(2, 27)$ over $2^{\bullet}(13 : 2)$, $PSL(2, 8)$ over $(9 : 2)$, and $PSL(3, 3)$ over $(13 : 3)$. We will also give our techniques of finding finite homomorphic images and their isomorphism images.

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In Memory of a special person whose watching over me from the sky,

Abuelita Carmen Luna(R.I.P)

Table of Contents

List of Tables

List of Figures

Introduction

A progenitor is defined as s a semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi\ w \mid \pi \in N, w$ a reduced word in the $t_i\}$, where 2^{*n} denotes the free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, ..., n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutionary generators. Thus, elements of N can be gathered on the left, every element of the progenitor can be represented as Pw, where $P \in N$ and w is a word in the symmetric generators. Indeed this representation is unique provided w is simplified so that adjacent symmetric generators are distinct. Thus any additional relation by which we must factor the progenitor must have the form $Pw(t_0, t_1, ..., t_{n-1})$. Let N be a group of permutations on n letters. $2^{*n} : N$ means 2^{*n} extended by N acting as automorphisms (by conjugations). The objective here is to factor the progenitor by relations, that equate elements of N to the product of t_is, that give finite homomorphic images.Once we have found a finite group as an image of the progenitor, we can find the Isomorphism Type of any finite group.

In Chapter 1, we list the definitions, theorems, and lemmas we used through the rest of the chapters. In Chapter 2 we used the following methods to insert additional relations to progenitors, first order relations, wreath product, and Curtis lemma. In Chapter 3, we give the following examples to explain the process of determining the isomorphism types of the extensions of finite groups, $2^{\bullet}S_z(8), 13^2 : (4 \times 2)$, $S_4 \times 2$ and $2^4 :^{\bullet} S_5$. In Chapter 4 we construct double coset enumeration of finite groups, $PGL_2(7)$ over S_4 and $PGL_2(13)$ over S_4 given the permutation representation of finite groups, and finding their generators. Similarly, in Chapter 5 we construct a double coset enumeration of $2^{\bullet}S_z(8)$ and factored by its center. In Chapter 6 we construct double coset enumeration over maximal subgroup of finite groups $2 \times PGL_2(27)$ over

$M = (13:2)$, $PSL_3(3)$ over $M = (13:3)$, and $PSL_2(8)$ over $M = (9:2)$. In Chapter 7 and Chapter 8 we used monomial progenitor to seek finite homomorphic images, but for some monomial progenitors no images were given due to MAGMA resources. In Chapter 9 we show how to compute and find the composition factors of large homomorphic images of G. Lastly, in Chapter 10, we organized our findings of finite homomorphic images using tables.

Chapter 1

Definitions, Theorems, and Lemmas

1.1 Preliminaries

1.1.1 Definitions

Definition 1.1. *A* **group** G $(G, *)$ *is a nonempty collection of elements with an associative operation* $*$*, such that:*

- *there exists an identity element,* $e \in G$ *such that* $e * a = a * e$ *for all* $a \in G$*;*
- *for every* $a \in G$*, there exists an element* $b \in G$ *such that* $a * b = e = b * a$*. [Rot95]*

Definition 1.2. *Let* G *be a set. A (binary)* **operation** *on* G *is a function that assigns each ordered pair of elements of* G *an element on* G*. [Rot95]*

Definition 1.3. *For group* G*, a* **subgroup** S *of* G *is a nonempty subset where* $s \in G$ *implies* $s^{-1} \in G$ *and* $s, t \in G$ *imply* $st \in G$*. We denote subgroup* S *of* G *as* $S \leq G$*. [Rot95]*

Definition 1.4. *Let* H *be a subgroup of group* G*.* H *is a* **proper** *subgroup of* G *if* $H \neq G$*. We denote this as* $H < G$*. [Rot95]*

Definition 1.5. *A* **symmetric group***,* S_X*, is the group of all permutations of* X*, where* $X \in \mathbb{N}$*.* S_X *is a group under compositions. [Rot95]*

Definition 1.6. *If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$. [Rot95]*

Definition 1.7. *A* **semigroup** $(G, *)$ *is a nonempty set G equipped with an associative operation. [Rot95]*

Definition 1.8. *If $x \in X$ and $\phi \in S_X$, then ϕ* **fixes** *x if $\phi(x) = x$ and ϕ* **moves** *x if $\phi(x) \neq x$. [Rot95]*

Definition 1.9. *For permutations $\alpha, \beta \in S_X$, α and β are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(a) = a \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y. \text{ [Rot95]}$$

Definition 1.10. *A permutation which interchanges a pair of elements is a* **transposition**. *[Rot95]*

Definition 1.11. *In group G, if a,b $\in G$, a and b* **commute** *if $a * b = b * a$. [Rot95]*

Definition 1.12. *A group G is* **abelian** *if every pair of elements in G commutes with one another. [Rot95]*

Definition 1.13. *Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G. [Rot95]*

Definition 1.14. *Let G be a group and $T = t_1, t_2, ..., t_n$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the* **progenitor** *to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_n. [Cur07]*

Definition 1.15. *Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by $N_G(H) = \{a \in G | aHa^{-1} = H\}$. [Rot95]*

Definition 1.16. *Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}. \text{ [Rot95]}$$

Definition 1.17. *Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we say G is* **elementary abelian**. *[Rot95]*

Definition 1.18. *Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \to H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all a,b $\in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to H, $G \cong H$, if there is exists an isomorphism $f : G \to H$. [Rot95]*

Definition 1.19. *Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *. [Rot95]*

Definition 1.20. *Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X. [Rot95]*

Definition 1.21. *Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1} a g | g \in G\}$. [Rot95]*

Definition 1.22. *The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$. [Rot95]*

Definition 1.23. *A* **general linear group**, *$GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$. [Rot95]*

Definition 1.24. *A* **special linear group**, *$SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$. [Rot95]*

Definition 1.25. *A* **projective special linear group**, *$PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}. \text{ [Rot95]}$$

Definition 1.26. *A* **projective general linear group**, *$PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}. \text{ [Rot95]}$$

Definition 1.27. (**Monomial Character**) *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H. [Led87]*

Definition 1.28. (**Character**) *Let $A(x) = (A_{ij}(x))$ be a matrix representation of G of degree m. We consider the character polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the **character** *of $A(x)$. [Led87]*

Definition 1.29. *The sun of squares of the degrees of the s=distinct irreducible characters of G is equal to $|G|$. The* **degree of a character** *χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character. [Led87]*

Definition 1.30. (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x)$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. Then lifting process preserves irreducibility. [Led87]*

Definition 1.31. (**Induced Character**) *Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then*

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G. [Led87]

Definition 1.32. *Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let C_α, $\alpha = 1, 2, ..., m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, 3, ..., m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \alpha = 1, 2, 3, ..., m. [Led87]$$

Definition 1.33. *Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$. [Rot95]*

Definition 1.34. *Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \lhd G$ to denote K as being normal in G. [Rot95]*

Definition 1.35. *Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St. [Rot95]*

Definition 1.36. *Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G. [Rot95]*

Definition 1.37. *Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$. [Rot95]*

Definition 1.38. *Let N be a group. The* **point stabilizer** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s. [Rot95]}$$

Definition 1.39. *Let N be a group. The* **coset stabilizer** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s. [Rot95]}$$

Definition 1.40. *Let G be a group. The* **center** *of G, $Z(G)$, is the set of all elements in G that commute with all elements of G. [Rot95]*

Definition 1.41. *A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1, \pi_2\omega_2, \dots}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1\omega_1, \pi_2\omega_2, \dots$ have been factored out. [Led87]

Definition 1.42. *We defined*

$$\mathcal{N}^i = C_{\mathcal{N}}(t_i); \mathcal{N}^{ij} = C_{\mathcal{N}}(\langle t_i, t_j \rangle) \text{ etc,}$$

single point and two point stabilizer in $\mathcal{N}$ respectively. The coset stabilizing subgroup, $\mathcal{N}^{(w)}$, of $\mathcal{N}$ is given by

$$\mathcal{N}^{(w)} = \pi \in \mathcal{N} : \mathcal{N}w\pi = \mathcal{N}w,$$

for w a word in the symmetric generators. Clearly $\mathcal{N}^w \leq \mathcal{N}^{(w)}$, and the number of cosets in the double coset $[w] = \mathcal{N}w\mathcal{N}$ is given by $|\mathcal{N}|/|\mathcal{N}^{(w)}|$, since $\mathcal{N}w\pi_1 \neq \mathcal{N}w\pi_2$
$\iff \mathcal{N}w\pi_1\pi_2^{-1} \neq \mathcal{N}w$
$\iff \pi_1\pi_2 \notin \mathcal{N}^{(w)}$
$\iff \mathcal{N}^{(w)}\pi_1\pi_2^{-1} \neq \mathcal{N}^{(w)}$
$\iff \mathcal{N}^{(w)}\pi_1 \neq \mathcal{N}^{(w)}\pi_2.$

Double Coset Enumeration Arithmetic

In order to obtain the index of $\mathcal{N}$ in $\mathcal{G}$ we shall perform a manual double coset enumeration of $\mathcal{G}$ over $\mathcal{N}$; thus we must find all double cosets $[w]$ and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $\mathcal{N}^{(w)}$ on the symmetric generators. We need only identify, for each $[w]$, the double coset to which $\mathcal{N}wt_i$ belongs for one symmetric generator t_i from each orbit. [Cur07]

Definition 1.43. First Isomorphism Theorem(F.I.T). *Let $\phi : G \to H$ is a homomorphism with $Ker\phi$. Then,*

$$\bullet Ker\phi \trianglelefteq G$$
$$\bullet G/Ker\phi \cong img\phi \text{ [Rot95]}$$

1.1.2 Theorems

Theorem 1.44. *The number of irreducible character of G is equal to the number of conjugacy classes of G. [Cur07]*

Theorem 1.45. *Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$. [Rot95]*

Theorem 1.46. *Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]*

Theorem 1.47. *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles. [Rot95]*

Theorem 1.48. *Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*
- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*
- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$. [Rot95]*

Theorem 1.49. *The intersection of any family of subgroups of a group G is again a subgroup of G. [Rot95]*

Theorem 1.50. *If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 1.51. *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. [Rot95]*

Theorem 1.52. *If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|. \text{ [Rot95]}$$

Theorem 1.53. *If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$. [Rot95]*

Theorem 1.54. *The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$. [Rot95]*

Theorem 1.55. *Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$. [Rot95]*

Theorem 1.56. *If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite. [Rot95]

Theorem 1.57. *If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$. [Rot95]*

Theorem 1.58. *Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n. [Rot95]*

Theorem 1.59. *If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker \rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H. [Rot95]*

Theorem 1.60. *If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set. [Rot95]*

Theorem 1.61. *Every two composition series of a group G are equivalent.*

We will refer to this Theorem as the **Jordan-Hölder Theorem***. [Rot95]*

Theorem 1.62. *Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$. [Rot95]*

1.1.3 Lemmas

Lemma 1.63. *Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*
- *If $H \lhd G$, then the subsets Hx are blocks of X. [Rot95]*

1.2 Construct the Character Table of S_6

In this section we used the **Frobinius** method to construct a character table for the symmetric group, S_6. Since we have the number of irreducible character or the number of partitions we were able to find the conjugacy classes of each partition.
The conjugacy classes of each partition is listed below.

Table 1.1: Conjugacy of Partitions

	χ	Conjugate	Class
1	[6]	$\longrightarrow$	(1^6)
2	[5 1]	$\longrightarrow$	$(1^4\ 2)$
3	[4 2]	$\longrightarrow$	$(1^2\ 2^2)$
4	$[4\ 1^2]$	$\longrightarrow$	$(1^3\ 3)$
5	$[3^2]$	$\longrightarrow$	(2^3)
6	[3 2 1]	$\longrightarrow$	(1 2 3)
7	$[3\ 1^3]$	$\longrightarrow$	$(1^2\ 4)$
8	$[2^3]$	$\longrightarrow$	(3^2)
9	$[2^2\ 1^2]$	$\longrightarrow$	(2 4)
10	$[2\ 1^4]$10	$\longrightarrow$	(1 5)
11	$[1^6]$	$\longrightarrow$	(6)

Since we have the list of conjugacy class of each partition now we create the the character table of S_6, given each characters with the terms of partitions in ascending order. Note the partition of each class is denoted as $(1^2 4)$ where we have the product of 2 1-cycles and one 4-cycles. Continue using the same procedure, the classes of S_6 and their sizes are listed on the following table.

Table 1.2: Character Table of S_6

α	(1^6)	$(1^4\ 2)$	$(1^3\ 3)$	$(1^2\ 4)$	$(1^2\ 2^2)$	(1 2 3)	(1 5)	(2^3)	(2 4)	(6)	(3^2)
h_α	1	15	40	90	45	120	144	15	90	120	40

A symmetric group, S_m, is expressed in terms of w_m where $m \leq 6$.
In Tables 2.2 and 4.1, pp. 50 and 106 we find that symmetric functions are:

$$\vdots$$

$$w_{-1} = 0$$

$$w_0 = 1$$

$$w_1 = s1$$

$$w_2 = \frac{1}{2}(s_1^2 + s_2)$$

$$w_3 = \frac{1}{6}(s_1^3 + 3s_1s_2 + 2s_3)$$

$$w_4 = \frac{1}{24}(s_1^4 + 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 + 6s_4)$$

$$w_5 = \frac{1}{120}(s_1^5 + 10s_1^3s_2 + 20s_1^2s_3 + 15s_1s_2^2 + 30s_1s_4 + 20s_2s_3 + 24s_5)$$

$$w_6 = \frac{1}{720}(s_1^6 + 15s_1^4s_2 + 40s_1^3s_3 + 45s_1^2s_2^2 + 144s_1s_5 + 120s_1s_2s_3 + 40s_3^2 + 15s_2^3 + 90s_2s_4$$
$$+120s_6 + 90s_1^2s_4)$$

Since the character of [6] is a homomorphic image that implies $F^{[6]} = \chi^{[6]} = 1$. Now we are going to apply the $SchurFunction$ to calculate the characters of χ.
We start with [5 1]

$$F^{[5\ 1]} = \begin{bmatrix} w_5 & w_6 \\ w_0 & w_1 \end{bmatrix} = w_5w_1 - w_6,$$

$s_1(\frac{1}{120}(s_1^5 + 10s_1^3s_2 + 20s_1^2s_3 + 15s_1s_2^2 + 30s_1s_4 + 20s_2s_3 + 24s_5)) - \frac{1}{720}(s_1^6 + 15s_1^4s_2 + 40s_1^3s_3 + 45s_1^2s_2^2 + 144s_1s_5 + 120s_1s_2s_3 + 40s_3^2 + 15s_2^3 + 90s_2s_4 + 120s_6 + 90s_1^2s_4)$
$= (\frac{1}{720}(6s_1^6 + 60s_1^4s_2 + 120s_1^3s_3 + 90s_1^2s_2^2 + 100s_1^2s_4 + 120s_1s_2s_3 + 144s_1s_5 - s_1^6 - 15s_1^4s_2 - 40s_1^3s_3 - 45s_1^2s_2^2 - 144s_1s_5 - 120s_1s_2s_3 - 40s_3^2 - 15s_2^3 - 90s_2s_4 - 120s_6 - 90s_1^2s_4)$
$= \frac{1}{720}(5s_1^6 + 45s_1^4s_2 + 80s_1^3s_3 + 45s_1^2s_2^2 + 90s_1^2s_4 + 0s_1s_2s_3 + 0s_1s_5 - 40s_3^2 - 15s_2^3 - 90s_2s_4 - 120s_6)$
Thus, we divide $\frac{F^{[5\ 1]}}{S_6}$ to get $(5, 3, 2, 1, 1, 0, 0, -1, -1, -1, -1)$.
Therefore, $\chi^{[5\ 1]} : (5, 3, 2, 1, 1, 0, 0, -1, -1, -1, -1)$.

Similarly, the rest of the characters were simplified as follow:

$$F^{[4\ 2]} = \begin{bmatrix} w_4 & w_5 \\ w_1 & w_2 \end{bmatrix} = w_4w_2 - w_5w_1$$

$= \frac{1}{720}(9s_1^6 + 45s_1^4s_2 + 0s_1^3s_3 + 45s_1^2s_2^2 - 90s_1^2s_4 + 0s_1s_2s_3 + 45s_2^3 + 90s_2s_4 - 144s_1s_5)$

$\chi^{[4\ 2]} : (9, 3, 0, -1, 1, 0, -1, 3, 1, 0, 0)$,

$$F^{[4\ 1^2]} = \begin{bmatrix} w_4 & w_5 & w_6 \\ w_0 & w_1 & w_2 \\ w_{-1} & w_0 & w_1 \end{bmatrix} = w_4(s_1^2 - w_2) - 1(s_1w_5 - w_6)$$

$= \frac{1}{720}(10s_1^6 + 30s_1^4s_2 + 40s_1^3s_3 - 90s_1^2s_2 + 0s_1^2s_4 - 120s_1s_2s_3 + 0s_1s_5 - 30s_2^3 + 0s_2^2s_4 + 40s_3^2 + 120s_6)$

$\chi^{[4\ 1^2]} : (10, 2, 1, 0, -2, -1, 0, -2, 0, 1, 1)$,

$$F^{[3^2]} = \begin{bmatrix} w_3 & w_4 \\ w_2 & w_3 \end{bmatrix} = w_3^2 - w_4w_2$$

$= \frac{1}{720}(5s_1^6 + 15s_1^4s_2 + 45s_1^2s_2^2 - 40s_1^3s_3 + 120s_1s_2s_3 + 80s_3^2 - 90s_1^2s_4 - 45s_2^3 - 90s_2s_4)$

$\chi^{[3^2]} : (5, 1, -1, -1, 1, 1, 0, -3, -1, 0, 2)$, and

$$F^{[3\ 2\ 1]} = \begin{bmatrix} w_3 & w_4 & w_5 \\ w_1 & w_2 & w_3 \\ w_{-1} & w_0 & w_1 \end{bmatrix} = -(w_3^2 - s_1w_5) + s_1(w_2w_3 - s_1w_4)$$

$\frac{1}{720}(16s_1^6 0s_1^4s_2 - 8 - s_1^3s_3 + 0s_1s_2^2 - 180s_1^2s_4 + 0s_1s_2s_3 + 144s_1s_5 + 0s_2s_3 + 0s_2s_4 - 80s_3^2 + 0s_6)$

$\chi^{[3\ 2\ 1]} : (16, 0, -2, 0, 0, 0, 1, 0, 0, 0, -2)$.

Thus, the results for each character is given in the character table of S_6.

Note, the character of $[1^6]$ is an alternative group so the character will be generated by using:

$$F^{[1^6]} = \begin{cases} 1 & \text{for} \quad number\ of\ even\ 2 - cycles \\ -1 & \text{for} \quad number\ of\ odd\ 2 - cycles \end{cases}$$

Table 1.3: Character Table of S_6

α	(1^6)	$(1^4\ 2)$	$(1^3\ 3)$	$(1^2\ 4)$	$(1^2\ 2^2)$	$(1\ 2\ 3)$	$(1\ 5)$	(2^3)	$(2\ 4)$	(6)	(3^2)
h_α	1	15	40	90	45	120	144	15	90	120	40
$\chi[6]$	1	1	1	1	1	1	1	1	1	1	1
$\chi[5\ 1]$	5	3	2	1	1	0	0	-1	-1	-1	-1
$\chi[4\ 2]$	9	3	0	-1	1	0	-1	3	1	0	0
$\chi[4\ 1^2]$	10	2	1	0	-2	-1	0	-2	0	1	1
$\chi[3^2]$	5	1	-1	-1	1	1	0	-3	-1	0	2
$\chi[3\ 2\ 1]$	16	0	-2	0	0	0	1	0	0	0	-2
$\chi[3\ 1^3]$											
$\chi[2^3]$											
$\chi[2^2\ 1^2]$											
$\chi[2\ 1^4]$											
$\chi[1^6]$											

For instance, the class (1 2 3) where the members are the product of one 1-cycle, two 2-cycle, and one 3-cycle. Let the (a) be 1-cycle, $(b, c)\&(d, e)$ 2-cycle and (f, g, h) 3-cycle. Note (f, g, h) is made of two 2-cycle, $(f, h)\&(f, g)$. Thus, counting all 2-cycles we conclude that there are in total four 2-cycles. Hence $(1\ 2\ 3) = 1$.
Therefore, $\chi^{[1^6]} : (1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1)$.

To complete the character table of S_6 we are going to find remaining characters of χ by using the conjugate of each partitions. For example we have $[3\ 1^3]$ is conjugate to $[4\ 1^2]$, $[5\ 1]$ is conjugate to $[2\ 1^4]$, $[3^2]$ is conjugate to $[2^3]$, and $[4\ 2]$ is conjugate to $[2^2\ 1^2]$.

Hence:

$\chi^{[3\ 1^3]} = \chi^{[4\ 1^2]}\chi^{[1^6]} = (10(1), 2(-1), 1(1), 0(-1), -2(1), -1(-1),$
$0(1), -2(-1), 0(1), 1(-1), 1(1))$
$\chi^{[3\ 1^3]} : (10, -2, 1, 0, -2, 1, 0, 2, 0, -1, 1),$

$\chi^{[2\ 1^4]} = \chi^{[5\ 1]}\chi^{[1^6]} = (5(1), 3(-1), 2(1), 1(-1), 1(1), 0(-1), 0(1), -1(-1),$
$-1(1), -1(-1), -1(1))$
$\chi^{[2\ 1^4]} : (5, -3, 2, -1, 1, 0, 0, 1, -1, 1, -1),$

$$\chi^{[2^3]} = \chi^{[3^2]}\chi^{1^6]} = (5(1), 1(-1), -1(1), -1(-1), 1(1), 1(-1), 0(1),$$

$- 3(-1), -1(1), 0(-1), 2(1))$

$\chi^{[2^3]} : (5, -1, -1, 1, 1, -1, 0, 3, -1, 0, 2),$

and

$$\chi^{[2^2\ 1^2]} = \chi^{[4\ 2]}\chi^{[1^6]} = (9(1), 3(-1), 0(1), -1(-1), 1(1), 0(-1), -1(1))$$

$3(-1), 1(1), 0(-1), 0(1))$

$\chi^{[2^2\ 1^2]} : (9, -3, 0, 1, 1, 0, -1, -3, 1, 0, 0)$

Therefore, we have completed the character table of S_6.

Table 1.4: Character Table of S_6

α	(1^6)	$(1^4\ 2)$	$(1^3\ 3)$	$(1^2\ 4)$	$(1^2\ 2^2)$	$(1\ 2\ 3)$	$(1\ 5)$	(2^3)	$(2\ 4)$	(6)	(3^2)
h_α	1	15	40	90	45	120	144	15	90	120	40
$\chi[6]$	1	1	1	1	1	1	1	1	1	1	1
$\chi[5\ 1]$	5	3	2	1	1	0	0	-1	-1	-1	-1
$\chi[4\ 2]$	9	3	0	-1	1	0	-1	3	1	0	0
$\chi[4\ 1^2]$	10	2	1	0	-2	-1	0	-2	0	1	1
$\chi[3^2]$	5	1	-1	-1	1	1	0	-3	-1	0	2
$\chi[3\ 2\ 1]$	16	0	-2	0	0	0	1	0	0	0	-2
$\chi[3\ 1^3]$	10	-2	1	0	-2	1	0	2	0	-1	1
$\chi[2^3]$	5	-1	-1	1	1	-1	0	3	-1	0	2
$\chi[2^2\ 1^2]$	9	-3	0	1	1	0	-1	-3	1	0	0
$\chi[2\ 1^4]$	5	-3	2	-1	1	0	0	1	-1	1	-1
$\chi[1^6]$	1	-1	1	-1	1	-1	1	-1	1	-1	1

Chapter 2

Methods on Finding Progenitors

2.1 Permutation Progenitors

Before factoring the progenitor $m^{*n} : N$, where m is the order of $t's$ and n is the number of $t's$, and N is the control group, by first order relation we are going to write a presentation of permutation progenitor. Since the progenitor $m^{*n} : N$ is infinite we write a permutation progenitor where we take N to be transitive in n letters. So we have a general form of a permutation progenitor in the following form:

$$\langle x, y, t | < x, y > \cong N, t^m, (t, N^i) \rangle, \text{where } N^i \text{ is the stabiliser of } i \text{ in } N$$

Since t commutes with the stabiliser of i in N, (t, N^i), we can obtain the number of conjugates of t. Using the definition we have that $[G : C_G(a)]$ is the number of conjugates of H in G. So to find the index of the centraliser of N and t also denoted as $Centraliser(N, t)$, we are going to calculate $[N : C_N(a)]$. Note that the index of the $Centraliser(N, t)$ is equal to the number of conjugates of t and also equal to the stabiliser of a single point in N. Applying this concept we are going to find the permutation progenitor of the following example, $2^{*13} : (13 : 4)$.

EXAMPLE: We are going to show how to write a permutation progenitor. This can be illustrated through a brief example of an infinite progenitor $2^{*13} : (13 : 4)$. Note, the control group $N = (13 : 4)$ is transitive in 13 letters and $(13 : 4) =$

$\langle(1,2,3,4,5,6,7,8,9,10,11,12,13),(1,8,12,5)(2,3,11,10)(4,6,9,7)\rangle$ where the generators of N are $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13)$ and $y \sim (1,8,12,5)(2,3,11,10)(4,6,9,7)$. Then the presentation of $(13:4)$ is

$$\langle x,y|y^4,y^{-2}x^{-1}b^2x^{-1},y^{-1}x^{-3}yx^{-2}\rangle$$

Now we let t be a symmetric generator, where $t \sim t_1$, and be of order 2. Since we let t to be t_1 we compute the stabiliser of the single point 1 in N, N^1. So $N^1 = \langle(2,9,13,6)(3,4,12,11)(5,7,10,8)\rangle$. Note that $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13)$ and $y \sim (1,8,12,5)(2,3,11,10)(4,6,9,7)$, if we take the generator y and conjugate by the generator x, then we get $y^x = (2,9,13,6)(3,4,12,11)(5,7,10,8)$. Since we stabilised 1 as a single point in N and t, to be the symmetric generator then we write $(t,N^i) = (t,y^x)$. Thus, the permutation progenitor of the group $2^{*13}:(13:4)$ is given as follows:

$$\langle x,y,t|y^4,y^{-2}x^{-1}y^2x^{-1},y^{-1}x^{-3}yx^{-2},$$

$$t^2,$$

$$(t,y^x)\rangle$$

In the next chapters we are going to apply this procedure to find permutation progenitors for every progenitor of the form $m^{*n}:N$.

2.2 Factoring $m^*:N$ by the First Order Relations

In order to factor the progenitor, $m^{*n}:N$, by all the first order relations, first, we compute the conjugacy classes of our group N. Then we compute the centraliser of representatives of each non-identity class. Lastly, we determine the orbits for each representative. The detailed work for factoring a progenitor by the first order relations is shown in this section.

EXAMPLE :

In this example we are going to focus on the previous progenitor $2^{*13}:(13:4)$ where N is $13:4$. Since we have a permutation progenitor which is an infinite group, we may obtain a finite group by factoring $2^{*13}:(13:4)$ by first order relations. We used the following codes in MAGMA.

```
>S:=Sym(13);
>xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
>yy:=S!(1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7);
>N:=sub<S|aa,bb>;
>#N;
>FPGroup(N);
Finitely presented group on 2 generators
Relations
    $.2^4 = Id($)
    $.2^-2 * $.1^-1 * $.2^2 * $.1^-1 = Id($)
    $.2^-1 * $.1^-3 * $.2 * $.1^-2 = Id($)
```

We convert this in terms of x and y to get the presentation of $2^{*13} : (13 : 4)$.

$$\langle x, y | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^{-2} \rangle$$

Now that we have the presentation for the N we continue on finding all the first order relations to factor the progenitor with. We do this in MAGMA as follow.

```
>C:=Classes(N);
>#C;
7
>for i in [1..#C] do i, C[i][3];end for;
>for j in [2..7] do for i in [1..52] do
if ArrayP[i] eq C[j][3] then C[j][3],Sch[i];
end if;end for;end for;
>for i in [2..7] do i, Orbits(Centraliser(N,C[i][3]));
end for;
```

This information is given in the following table.

Now, to find all first order relations we use the information listed in Table 1.1. We take the representative from each class C_2, C_3, C_4, C_5, C_6, and C_7 and multiply it on the right by a representative from each orbit until we have exhausted all seven classes. Noticed C_1 is not included since it is identity. Before we give all possible first order relations we show the procedure for one of the classes. For instance, in C_2 the representative for this class is $(xy)^2$ and the representative of the first orbit is 10, so we have $(xy)^2t$ where $t \sim t_1$. Now we find a permutation in terms of words that conjugates t_1 to given t_{10} and that is x^{-4}. So we get the first relation to be $((xy)^2t^{x^{-4}})$. Following

Table 2.1: Conjugacy Classes of $N = 13 : 4$

Class	Representative of the class	# of elements in the class	Orbits
C_1	Identity	1	{1},{2},{3},{4}, {5},{6},...,{13}
C_2	$(xy)^2$=(1,6) (2,5) (3,4) (7,13) (8,12) (9,11)	4	{10},$\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$, $\{7, 13\}$, $\{8, 12\}$, $\{9, 11\}$
C_3	xy=(1, 3, 6, 4)(2, 11, 5, 9)(7, 12, 13, 8)	4	{10},{1,3,6,4}, {2,11,5,9}, {7,12,13,8}
C_4	$y^{-1}x^{-1}$=(1, 4, 6, 3)(2, 9, 5, 11)(7, 8, 13, 12)	4	{10},{1,4,6,3}, {2,9,5,11}, {7,8,13,2}
C_5	x=(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)	13	{1, 2, 3, 4, 5, 6, 7,8,9,...,13}
C_6	x^2=(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)	13	{1,3,5,7,9,11, 13,2,4,6,8,10,12}
C_7	x^4=(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)	13	{1,5,9,13,4,8,12, 3,7,11,2,6,10}

the same process we are able to obtain all possible first order relations which are: $((xy)^2 * t^{a^{-4}})^c, ((xy)^2t)^d, ((xy)^2t^x)^e, ((xy)^2t^{x^6})^f, ((xy)t^{x^{-4}})^g, ((xy)t)^h, ((xy)t^x)^i,$ $((xy)t^{x^6})^j, ((y^{-1}x^{-1})t^{x^{-4}})^k, ((y^{-1}x^{-1})t)^l, ((y^{-1}x^{-1})t^x)^m, ((y^{-1}x^{-1})t^{x^6})^n, (xt)^o,$ $(x^2t)^p, (x^4t)^q$

Hence, to get finite images we factor the permutation progenitor of $2^{*13} : (13 : 4)$ by all of the first order relations. Then we have the following presentation of the progenitor factored by every single first order relation:

$$\langle x, y, t | y^4, y^{-2}x^{-1}b^2x^{-1}, y^{-1}x^{-3}yx^{-2},$$

$$t^2,$$

$$(t, y^x), ((xy)^2 * t^{a^{-4}})^c, ((xy)^2t)^d,$$

$$((xy)^2t^x)^e, ((xy)^2t^{x^6})^f, ((xy)t^{x^{-4}})^g, ((xy)t)^h, ((xy)t^x)^i,$$

$$((xy)t^{x^6})^j, ((y^{-1}x^{-1})t^{x^{-4}})^k, ((y^{-1}x^{-1})t)^l, ((y^{-1}x^{-1})t^x)^m,$$

$$((y^{-1}x^{-1})t^{x^6})^n, (xt)^o, (x^2t)^p, (x^4t)^q\rangle$$

The following table shows some of the finite homomorphic images that were obtained by factoring the progenitor by first order relations.

Table 2.2: Primitive Group (13,4)$\cong 2^{*13} : (13 : 4)$

$\#G$	c	d	...	l	m	n	o	p	q	*Isomorphism Type*
104	0	0	...	0	0	0	0	0	2	C_2
1352	0	0	...	0	0	0	4	0	0	$13^2 : (4 \times 2)$
58240	0	0	...	0	0	5	0	5	0	$2 \times S_z(8)$
29120	0	0	...	0	5	0	5	0	7	$S_z(8)$

2.3 Wreath Product Progenitors

The **Wreath Product** is a semi-direct product of two groups. Let X and Y be sets such $X \cap Y = \emptyset$. Define $H \leq SX$ and $K \leq SY$.
Let $\mathbb{Z} = X \times Y$, such that $X \cap Y = \emptyset$. Define a permutation group $\mathbb{Z}$. Let $\gamma \in H, y \in Y$, and $k \in K$, where

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma) \\ (x, y_1) \mapsto (x, y_1) \quad \text{if } y \neq y_1. \end{cases}$$

Wreath Product of Groups: Let H be a permutation group on X and K be a permutation group on Y.
Define $Z = X \times Y = \{(x, y)|x \in X, y \in Y\}$.
Construction of a permutation group on Z, (Called the Wreath Product of H by K)

$$H \wr k.$$

2.3.1 Writing Wreath Product Progenitor of Groups

We now construct a presentation for a wreath product progenitor. The process it takes to create a wreath product progenitor is by taking two known groups with their generators and presentations. The example listed below describes the process on how to construct a wreath product progenitor.

2.3.2 Writing a Wreath Product Progenitor of $2^{*30} : ((5^3 : 2^3) : 3)$ on $\mathbb{Z}_{10} \wr \mathbb{Z}_3$.

Let $H =< (1,2,3,4,5,6,7,8,9,10) >$ and $K =< (11,12,13) >$.
Define $Z = X \times Y = \{(1,11),(2,11),(3,11),(4,11),(5,11)(6,11)(7,11)(8,11)(9,11)$ $(10,11)(1,12)(2,12)(3,12)(4,12)(5,12)(6,12)(7,12)(8,12)(9,12)(10,12)(1,13)(2,13)$ $(3,13)(4,13)(5,13)(6,13)(7,13)(8,13)(9,13)(10,13)\}$.
Now, we use the definition of the wreath product to conjugate each element of Z by each transversal in K, the tables below shows the mapping:

Table 2.3: $\gamma(11)$

1.	$(1,11)$	$\to$	(2,11)	(2)
2.	$(2,11)$	$\to$	(3,11)	(3)
3.	$(3,11)$	$\to$	(4,11)	(4)
4.	$(4,11)$	$\to$	(5,11)	(5)
5.	$(5,11)$	$\to$	(6,11)	(6)
6.	$(6,11)$	$\to$	(7,11)	(7)
7.	$(7,11)$	$\to$	(8,11)	(8)
8.	$(8,11)$	$\to$	(9,11)	(9)
9.	$(9,11)$	$\to$	(10,11)	(10)
10	$(10,11)$	$\to$	(1,11)	(1)

Table 2.4: $\gamma(12)$

11.	$(1,12)$	$\to$	(2,12)	(12)
12.	$(2,12)$	$\to$	(3,12)	(13)
13.	$(3,12)$	$\to$	(4,12)	(14)
14.	$(4,12)$	$\to$	(5,12)	(15)
15.	$(5,12)$	$\to$	(6,12)	(16)
16.	$(6,12)$	$\to$	(7,12)	(17)
17.	$(7,12)$	$\to$	(8,12)	(18)
18.	$(8,12)$	$\to$	(9,12)	(19)
19.	$(9,12)$	$\to$	(10,12)	(20)
20.	$(10,12)$	$\to$	(1,12)	(11)

Table 2.5: $\gamma(13)$

21.	$(1,13)$	$\to$	(2,13)	(22)
22.	$(2,13)$	$\to$	(3,13)	(23)
23.	$(3,13)$	$\to$	(4,13)	(24)
24.	$(4,13)$	$\to$	(5,13)	(25)
25.	$(5,13)$	$\to$	(6,13)	(26)
26.	$(6,13)$	$\to$	(7,13)	(27)
27.	$(7,13)$	$\to$	(8,13)	(28)
28.	$(8,13)$	$\to$	(9,13)	(29)
29.	$(9,13)$	$\to$	(10,13)	(30)
30	$(10,13)$	$\to$	(1,13)	(21)

Referring back to Table 2.3 we get 1 maps to 2, 2 maps to 3, 3 maps to 4,..., and 10 maps back to 1. Thus, we get the permutation $\gamma(11) = \{(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)\}$. We continue by applying the same procedure for Tables 2.4 and 2.5 to get the permutations of $\gamma(12) = \{(11, 12, 13, 14, 15, 16, 17, 18, 19, 20)\}$ and $\gamma(13) = \{(21, 22, 23, 24, 25, 26, 27, 28, 29, 30)\}$.

To get the last permutation we conjugate every element of $Z = \{(1, 11), (2, 11), (3, 11), (4, 11), (5, 11)(6, 11)(7, 11)(8, 11)(9, 11)(10, 11)(1, 12)(2, 12)(3, 12)(4, 12)(5, 12)(6, 12)(7, 12)(8, 12)(9, 12)(10, 12)(1, 13)(2, 13)(3, 13)(4, 13)(5, 13)(6, 13)(7, 13)(8, 13)(9, 13)(10, 13)\}$ by the permutation $K = (3, 4, 5)$

Table 2.6: k^*

1.	$(1, 11)$	$\to$	(1,12)	(11)	16.	$(6, 12)$	$\to$	$(6, 13)$	(26)
2.	$(2, 11)$	$\to$	(2,12)	(12)	17.	$(7, 12)$	$\to$	$(7, 13)$	(27)
3.	$(3, 11)$	$\to$	(3,12)	(13)	18.	$(8, 12)$	$\to$	$(8, 13)$	(28)
4.	$(4, 11)$	$\to$	(4,12)	(14)	19.	$(9, 12)$	$\to$	$(9, 13)$	(29)
5.	$(5, 11)$	$\to$	(5,12)	(15)	20.	$(10, 12)$	$\to$	$(10, 13)$	(30)
6.	$(6, 11)$	$\to$	(6,12)	(16)	21.	$(1, 13)$	$\to$	$(1, 11)$	(1)
7.	$(7, 11)$	$\to$	(7,12)	(17)	22.	$(2, 13)$	$\to$	$(2, 11)$	(2)
8.	$(8, 11)$	$\to$	(8,12)	(18)	23.	$(3, 13)$	$\to$	$(3, 11)$	(3)
9.	$(9, 11)$	$\to$	(9,12)	(19)	24.	$(4, 13)$	$\to$	$(4, 11)$	(4)
10.	$(10, 11)$	$\to$	(10,12)	(20)	25.	$(5, 13)$	$\to$	$(5, 11)$	(5)
11.	$(1, 12)$	$\to$	(1,13)	(21)	26.	$(6, 13)$	$\to$	$(6, 11)$	(6)
12.	$(2, 12)$	$\to$	(2,13)	(22)	27.	$(7, 13)$	$\to$	$(7, 11)$	(7)
13.	$(3, 12)$	$\to$	(3,13)	(23)	28.	$(8, 13)$	$\to$	$(8, 11)$	(8)
14.	$(4, 12)$	$\to$	(4,13)	(24)	29.	$(9, 13)$	$\to$	$(9, 11)$	(9)
15.	$(5, 12)$	$\to$	(5,13)	(25)	30.	$(10, 13)$	$\to$	$(10, 11)$	(10)

From Table 2.6 we can see that 1 maps to 11, 11 maps to 21, and 21 maps back to 1. Following the same procedure from above we get that the permutation of Z^K also denoted as $k^* = \{(1, 11, 21)(2, 12, 22)(3, 13, 23)(4, 14, 24)(5, 15, 25)(6, 16, 26)(7, 17, 27)(8, 18, 28)(9, 19, 29)(10, 20, 30)\}$. Thus, every generator of $\gamma(11)$, $\gamma(12)$, $\gamma(13)$, and K are placed together in the following form:

$$< a > \times < b > \times < c >:< k^* >$$

where $a = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$, $b = (11, 12, 13, 14, 15, 16, 17, 18, 19, 20)$,

$c = (21, 22, 23, 24, 25, 26, 27, 28, 29, 30)$, and $d = (1, 11, 21)(2, 12, 22)(3, 13, 23)(4, 14, 24)$ $(5, 15, 25)(6, 16, 26)(7, 17, 27)(8, 18, 28)(9, 19, 29)(10, 20, 30)$. Furthermore, we find the order of $a = 10$, $b = 10$, $c = 10$, and $k^* = 3$. Up to this point the first part of the representation should have a^{10}, b^{10}, c^{10}, and d^3. Following with the next step there is a direct-product between each generator which means that a, b, c, and d commutes with each other, (a, b),(a, c),and (b, c). We also find the semi-direct product action among a, b, c and k^*. We apply the definition of a semi-direct product to check what does d do to a, b, and c which we get $a^d = b$, $b^d = c$, and $c^d = a$. Therefore, a presentation is written for this groups in the form of $\mathbb{Z}_{10} \wr \mathbb{Z}_3 = a^{10}, b^{10}, c^{10}, d^3, (a, b), (a, c)(b, c), a^d = b, b^d = c, c^d = a >$.

Now that we have a presentation we verify if it is isomorphic to $(1000 : 3)$. To show this we are going to run a few loops in MAGMA.

```
>  H<a,b,c,d>:=Group<a,b,c,d|a^10,b^10,c^10,d^3, (a,b),
>   (a,c)(b,c),a^d=b,b^d=c,c^d=a>;
> #H;
3000
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> N:=TransitiveGroup(30,437);
> IsIsomorphic(N,H1);
true Mapping from: GrpPerm: N to GrpPerm: H1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: H
```

Hence, we have constructed and verified the presentation of the group N. Now we continue by finding the stabilizer of t_1, where $t \sim t_1$ and finishing the infinite progenitor by using the following code in MAGMA.

```
> S:=Sym(30);
> xx:=S!(1,2,3,4,5,6,7,8,9,10);
> yy:=S!(11,12,13,14,15,16,17,18,19,20);
> zz:=S!(21,22,23,24,25,26,27,28,29,30);
> ww:=S!(1,11,21)(2,12,22)(3,13,23)(4,14,24)
(5,15,25)(6,16,26)(7,17,27)(8,18,28)(9,19,29)
(10,20,30);
> N:=sub<S|xx,yy,zz,ww>;
> N eq sub<G|xx,yy,zz,ww>;
true
```

```
> N3:=Stabiliser(N,1);
> N3;
Permutation group N3 acting on a set of cardinality 30
Order = 100 = 2^2 * 5^2
    (11, 12, 13, 14, 15, 16, 17, 18, 19, 20)
    (21, 22, 23, 24, 25, 26, 27, 28, 29, 30)
```

After running the codes in MAGMA we were able to find the stabiliser of t_1 which is $\langle(11,12,13,14,15,16,17,18,19,20),(21,22,23,24,25,26,27,28,29,30)\rangle$. We store the two permutations as b and c. Now we have t commutes with b and c. Thus, t^2, (t,b) and (t,c) are added to the group N finalizing the infinite progenitor $2^{*30}:(\mathbb{Z}_{10}\wr\mathbb{Z}_3)$.

```
H<a,b,c,d,t>:=Group<a,b,c,d,t|a^10,b^10,c^10,d^3, (a,b),
(a,c)(b,c),a^d=b,b^d=c,c^d=a,
t^2,
(t,b),(t,c)>;
```

In conclusion we have obtained a presentation from $(\mathbb{Z}_{10}\wr\mathbb{Z}_3)$,
$H < a,b,c,d,t >:= Group < a,b,c,d,t|a^{10},b^{10},c^{10},d^3,(a,b),(a,c)(b,c),a^d = b,b^d = c,c^d = a,t^2,(t,b),(t,c) >$, using the wreath product. After having the presentation verified we continue by factoring by the first order relation.

2.3.3 $\mathbb{Z}_{10}\wr\mathbb{Z}_3$ Factor by First Order Relation

Giving the presentation representation of $\mathbb{Z}_{10}\wr\mathbb{Z}_3$ we are going to factor the presentation by first order relation. To find all of the first order relations, we are going to store the generators and label them as x,y,z, and w. We run the following codes in MAGMA

```
  S:=Sym(30);
xx:=S!(1,2,3,4,5,6,7,8,9,10);
yy:=S!(11,12,13,14,15,16,17,18,19,20);
zz:=S!(21,22,23,24,25,26,27,28,29,30);
ww:=S!(1,11,21)(2,12,22)(3,13,23)(4,14,24)(5,15,25)(6,16,26)
(7,17,27)(8,18,28)(9,19,29)(10,20,30);
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^10,y^10,z^10,w^3,(x,y),(x,z),
(y,z),x^-1*w^-1*z*w,y^-1*w^-1*x*w,
```

```
t^2,
(t,y),(t,z)>;
C:=Classes(N);
#C;
360
for i in [2..360] do i, Orbits(Centraliser(N,C[i][3]));
end for;
for j in [2..360] do for i in [1..13] do if ArrayP[i] eq
C[j][3] then C[j][3],Sch[i];
end if;
end for;
end for;
```

We use the process in section 2.2. First find every class, the representative and orbits from each class. Now take a representative from each class and right multiply by an element from each orbit ,until we have exhausted every single class. In this case since we have 360 relations we are going to add only a few relations. After multiplying representatives with orbits we get the following presentation representation of $\mathbb{Z}_{10} \wr \mathbb{Z}_3$.

```
for a,b,c,d,e,f,g in [0..10] do
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^10,y^10,z^10,w^3,(x,y),(x,z),
(y,z),x^-1*w^-1*z*w,y^-1*w^-1*x*w,t^2,(t,y),(t,z),(w*t)^a,
(w^-1*t)^b,(x^2*t)^c,(x*t)^d,(x^-1*t)^e,(x*y*t)^f,(x*w*t)^g>;
if #G gt 20 then #G,a,b,c,d,e,f,g;
end if; end for;
```

We use the loop listed above to obtained finite homomorphic images, see Chapter 10.

2.4 Curtis Lemma

2.4.1 Factoring by Curtis Lemma Relations

In order to find a finite homomorphic image we take a progenitor of the form $m^{*n} : N$ and factor by relations. However, finding simple groups factor by random relators can be difficult since we want to produce interesting groups.To find such images, Robert Curtis discovered a lemma where the elements of the control group of N can be written in terms of symmetric generators. In this section we are going to use Curtis lemma to generate symmetric presentations for progenitors to find homomorphic images.

We find the stabilizer of two elements say 1 and 2 and we determine the centralizer of the two elements.

Lemma 2.1. *(Curtis Lemma)*

$N\cap < t_i, t_j > \leq C_N(N^{ij})$ *where* N_{ij} *denotes the stabilizer in* N *of the two points* i *and* j*. [Rot95]*

Note:
If the $|t_i| = 2$, $|t_j| = 2$, and $|t_i t_j|$=n, then $< t_i, t_j >= D_{2n}$, the Dihedral group of order $2n$. We also know the center of D_{2n}:

$$Center(D_{2n}) = \begin{cases} 1, & \text{if n is odd} \\ < (t_i, t_j)^{\frac{n}{2}} >, & \text{if n is even.} \end{cases}$$

Lemma 2.2.

(i) If g *belongs to* N *and* $i^g = i$ *and* $j^g = j$ *then we should factor the progenitor by the relation* $(t_i t_j)^k = g$ *for any even positive integer* k*. .*

(ii) if g *belongs to* N *and* $i^g = j$ *and* $j^g = i$ *then we should factor the progenitor by the relation* $(g t_i)^k = 1$ *for any odd positive integer* k*.*
In other words we have:

$$= \begin{cases} (t_i t_j)^k = g & \textit{where k is even and fixes 1 and 2} \\ (g t_i)^k = 1 & \textit{where k is odd and g sends 1 to 2} \end{cases} [Rot95]$$

Example Factor $2^{*12} : S_4 \times 2$ by relations using the presentation of G:

$$< v, w, x, y, z | v^2, w^3, x^2, y^2, z^2, (w^{-1}v)^2, w^{-1}xwy, wxw^{-1}z, vxvy, (xy)^2 >$$

To produce a finite image we have to factor the presentation by relators. Some of the relations are obtained through the lemma. Let $t \sim t_1$. Now we find all elements that centraliser the subgroup $< t_1 >$. The point-stabiliser of 1 in N is given as a permutation group acting on a set of cardinality 12 of order 2. Now convert the permutations

$(2,7)(3,11)(4,5)(8,10)(9,12)$ and $(2,8)(3,9)(7,10)(11,12)$ into words to get:

$$N^1 = \{\langle(2,7)(3,11)(4,5)(8,10)(9,12),(2,8)(3,9)(7,10)(11,12)\rangle\} = \{\langle vxyzw^{-1}, x\rangle\}.$$

Then, use the relations to factor the progenitor, $2^{*12} : S_4 \times 2$. Keep in mind that the relations are obtained from stabilising t_1, so we have t_1 commute with $vxyzw^{-1}$ and x. So we have the following presentation:

$$\langle v, w, x, y, z, t | v^2, w^3, x^2, y^2, z^2, (w^{-1}v)^2, w^{-1}xwy, wxw^{-1}z, vxvy, (xy)^2,$$

$$t^2,$$

$$(t, vxyzw^{-1}), (t, x)\rangle.$$

Now, to find the centraliser of the point-stabiliser of two points we are going to apply the lemma. Let the point-stabiliser of the two elements 1 and 2 be denoted by $N^{t_1t_2}$.

Now we compute the centraliser of the stabiliser of point 1 and 2, $Centraliser(N, Stabiliser(N,[1,2]))$. The $Stabiliser(N,[1,2]) =< e >$ so now we compute the centraliser by finding which elements of N commutes with identity. So we check $Centraliser(N, Stabiliser(N,[1,2])) = \{e,(1,2)(3,6)(4,8)(5,9)(7,11)\}$. Using Curtis Lemma we write the following relation in the form :

$$((1,2)(3,6)(4,8)(5,9)(7,11)t_1)^k = 1, \text{ where k is odd}$$

We write the permutation $(1,2)(3,6)(4,8)(5,9)(7,11)$ in words so we have the relations $(vt_1)^k = 1$. Therefore the following presentation for the progenitor is listed below:

$$\langle v, w, x, y, z, t | v^2, w^3, x^2, y^2, z^2, (w^{-1}v)^2, w^{-1}xwy, wxw^{-1}z, vxvy, (xy)^2,$$

$$t^2,$$

$$(t, vxyzw^{-1}), (t, x), (vt)^k\rangle.$$

2.4.2 Factor $2^{*12} : S_4 \times 2$ by First Order Relation

Now we factor the presentation representation of $2^{*12} : S_4 \times 2$ by first order relation. Using Curtis Lemma we add $(vt)^k$ to the presentation the progenitor. Since we have

$$\langle v, w, x, y, z, t | v^2, w^3, x^2, y^2, z^2, (w^{-1}v)^2, w^{-1}xwy, wxw^{-1}z, vxvy, (xy)^2,$$

$$t^2,$$

$$(t, vxyzw^{-1}), (t, x), (vt)^k\rangle,$$

now we factor the presentation representation by first order relation.
To find the relators we first find the classes of N. Since the classes are given in terms of permutations then we change them to words. So the permutation representations are $xyz, x, xy, v, vwy, w, vx, vwyz$, and wy. Next, we determine the orbits of N. Then, we take a representative from each orbit and right multiply by the permutation representation to obtain the following relations $(xyzt)^a, (xt)^b, (xt^v)^c, (xyt^{(xw^2)})^d, (xyt)^e, (vt^{(xw^2)})^f$, $(vt^{(w^2y)})^g, (vt)^h, (vwyt^v)^i, (vwyt)^j, (vwyt)^k, (wt)^l, (wt^y)^m, (vxt^{(xw^2)})^n, (vxt)^o$, $(vwyzt^v)^p, (vwyzt)^q, (wyt)^r, (wyt^y)^s$. Therefore, we factor the presentation representation of $2^{*12} : S_4 \times 2$ by all of the first order relations. Thus, we get the following:

$$\langle v, w, x, y, z, t | v^2, w^3, x^2, y^2, z^2, (w^{-1}v)^2, w^{-1}xwy, wxw^-1z, vxvy, (xy)^2,$$

$$t^2,$$

$$(t, vxyzw^{-1}), (t, x), (vt)^{m1}, (xyzt)^a, (xt)^b, (xt^v)^c, (xyt^{(xw^2)})^d, (xyt)^e, (vt^{(xw^2)})^f,$$

$$(vt^{(w^2y)})^g, (vt)^h, (vwyt^v)^i, (vwyt)^j, (vwyt)^k, (wt)^l, (wt^y)^m, (vxt^{(xw^2)})^n, (vxt)^o,$$

$$(vwyzt^v)^p, (vwyzt)^q, (wyt)^r, (wyt^y)^s\rangle$$

A table with finite homomorphic images of $2^{*12} : S_4 \times 2$ is given on Chapter 10.

Chapter 3

Isomorphism Types of Some Groups

In the following Chapter a few examples are given and the process we use to solve extension problems of finite groups. To find an isomorphism type of a finite group a few codes were used in MAGMA to print out the following, the set of composition factors, NormalLattice, the largest abelian group, etc. Moreover, we write the composition series of the group. Keep in mind the composition series consist of simple factors and that every finite group has a composition series. The rest of the work we use to prove the isomorphism type of G is listed above.

3.1 Preliminaries

$$G = H_0 \geq H_1 \cdots H_m = 1$$

is a refinement of a normal series

$$G = H_0 \geq H_1 \cdots H_m = 1$$

if $G_0, G_1, ..., G_n$ is a subsequence of $H_0, H_1, ..., H_m$. A **composition series** is a normal series

$$G = G_0 \geq G_1 \cdots G_n = 1$$

in which, for all i either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_1$.
Jordan Hölder Every two composition series of a group G are equivalent.
If G has a composition series, then the factor groups of this series are called the **composition factors** of G.
If K $\geq$ G, then a (**right**) **transversal** of K in G is a subset T of G consisting of one element from each right coset of K in G. Now lets express how some groups are represented by many different extension. If K and Q are groups, then and **extension** of K by Q is a group G having a normal subgroup $K_1 \cong$ K with $G/K_1 \cong$ Q. If H and K are groups, then their **direct product**, denotes by H $\times$ K, is the group with elements all ordered pairs (h, k), where $h \in$ H and $k \in$ K and with operations

$$(h, k)(h', k') = (hh', kk')$$

A group G is a **semi** $-$ **direct product** of the subgroups K by the subgroups Q, denoted by G = K : Q, if K is normal in G and K has a complement $Q_1 \cong$ Q.
There are other two extensions we need to consider. For instance, A **central extension** of K by Q is an extension G of K by Q with K $\leq$ Z(G). And a **mixed extension** combines the properties of both a semi-direct product and central extension, where G = NK and N is a normal subgroup of a group G but is not central.

3.2 Minimal Permutation Representation

Now we find a minimal degree faithful permutation representation of $G1$ and investigate some properties of G. We want to use this to find isomorphism, maximal subgroup and use the more convenient representation of $G1$. To investigate these properties of G we are going to apply the following codes for each extension problems. For the following group we are going to find a minimal permutation representation and prove its isomorphism type.

```
S:=Sym(13);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=S!(1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,
t^2,
(t,y^x),((y^-1*x^-1)*t^(x^6))^5,
(x^2*t)^5>;
#G;
/*58240*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
G1;
/*Permutation group G1 acting on a set of cardinality 1120
Order = 58240 = 2^7 * 5 * 7 * 13*/
```

Note, the number or permutation on $G1$ is a set of cardinality 1120. Now, we are going to find a minimal permutation representation of G. To do so we are going to use MAGMA. MAGMA tells us the set of all subgroups on $G1$ and the subgroups that gives faithful permutation representations. For instance, the following code helps us determine the minimal degree using a faithful permutation representation.

```
SL:=Subgroups(G1);
T:={X`subgroup:X in SL};
TrivCore:={H:H in T|#Core(G1,H) eq 1};/
mdeg := Min({Index(G1,H):H in TrivCore});
mdeg;
/*1040*/
```

Now the minimum permutation representation of $G1$ decreased to 1040, so now we check in the set of H and find all the subgroups that have minimum degree.

```
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
/*1*/
H:= Rep(Good);
#H;
/*56*/
f,G1,K := CosetAction(G1,H);
G1;
```

Hence, there is only one faithful subgroup of order 56. Thus, the minimum permutation representation of group $G1$ is acting on a set of cardinality 1040. Now we prove the isomorphism type of G using a set of cardinality of 1040 instead of 1120.

3.3 Central Extensions

3.3.1 Prove Isomorphism Type, $2^{\bullet}S_z(8)$

In this section an example of a central extension is given. We are going to referred to a simple group $2 \cdot S_z(8)$. We know that a **central extension** of K by Q is an extension G of K by Q with K $\leq$ Z(G). Thus we check and verify the center, $Z(G)$, of the finite group $2 \cdot S_z(8)$. We start as follow:

Isomorphism type of G

Proof :

From the progenitor $2^{*13} : (13, 4)$ a finite homomorphic image and its composition factor is given below: $G < x, y, t >:= Group < x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^{-2},$

$t^2,$

$(t, y^x), ((y^{-1}x^{-1})t^{(x^6)})^5, (x^2t)^5 >; .$

```
>f,G1,k:=CosetAction(G,sub<G|x,y>);
>IN:=sub<G1|f(x),f(y)>;
>CompositionFactors(G1);
 G
    |  2B(2, 8)                     = Sz(8)
    *
    |  Cyclic(2)
    1.
```

Then, we have the composition factor of G.

We will now prove the isomorphism type of G.

So we write the composition series of G

$$G = G_0 \supseteq G_1 \supseteq G_2 \ where \ G_2 = 1.$$

Hence, the composition factors are:

$$G = (G_0/G_1)(G_1/G_2)$$
$$= (G_0/G_1)G_1$$
$$= Sz(8)C_2.$$

To verify the isomorphism type of the group we start by looking at the NormalLattice of G.

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------

[3]  Order 58240  Length 1  Maximal Subgroups: 2
---
[2]  Order 2      Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

Note that we have subgroup [2] with order 2 and Subgroup [3] with order 58240. In this case the largest abelian group can either be 2 or 58240. But in this case the largest abelian group is the subgroup [2], where $NL[2]$ is of order 2. Now we find G_1 where G_1 is an extension of G_0 by the factor of (G_0/G_1). Since $(G_0/G_1) = S_z(8)$ this implies that $G_1 = S_z(8)C_2$. So we find the extension of G_1. To do so we determine the order of $S_z(8)$ and C_2. Since the order of $S_z(8) = 29120$ and the order of $C_2 = 2$. By the definition of central extension, C_2 is the center of G_1. Thus, $NL[2]$ is the center of $NL[3]$. Therefore, $NL[3]$ is isomorphic to $2^{\bullet}S_z(8)$.
Now we prove that G_1 is isomorphic to $C_2^{\bullet}S_z(8)$
Since $NL[2]$ is the center of $NL[3]$ where $NL[3]$ is Q. Then we factor Q by the center.

```
>q,ff:=quo<G1|NL[2]>;
>q;
Permutation group q acting on a set of cardinality 560
```

```
Order = 29120 = 2^6 * 5 * 7 * 13
 >#q;
 29120
 >#Generators(q);
 >Generators(q);
```

Since there are three generators of q we store each generator as $q1, q2$, and $q3$.

```
q1:=q!(2,3,5,10,19,23,16,4,7,15,11,8,18)(6,13,28,59,...
q2:=q!(2,4,8,19)(3,5,11,15)(6,14,32,71)(7,16,10,23)...
q3:=q!(1,2)(3,6)(4,9)(5,12)(7,17)(8,20)(10,24)(11,25)...
```

And we check if q is isomorphic to $S_z(8)$.

```
>s:=IsIsomorphic(q,Sz(8));
>s;
true
```

Since q is isomorphic to $S_z(8)$ then we write a presentation of q using the code $FPGroup(q)$;.

```
 FPGroup(q);
<a^13,b^4,c^2,b^-2*a^-1*b^2*a^-1,
b^-1*a^-3*b*a^-2,a*c*a^-1*b^-2*a*c*b^2*a,
a*c*a^-1*b^-1*a*c*a^-1*b,b*c*b^-2*c*a*b^-2*c*b^2*c*b^-1*c,
c*a^-1*c*b^-1c*a^-2c*a*c*b^2*c*a,
c*a^-1*b^-2*c*b*c*a^-1*c*b^2*c*a*c*b^-1>
```

Now we find the transversals of G_1 and $NL[2]$. So we run the codes in Magma:

```
T:=Transversal(G1,NL[2]);
T[2];
T[3];
T[4];
```

Since the transversals are given in terms of permutations we are going to store the results as A, B, and C.
Likewise for the center, since $NL[2]$ is the center of $G1$. Thus, the center of $NL[2].2$ is stored as D.

```
D:=G1!(1,61)(2,209)(3,77)(4,248)(5,249)(6,250)(7,673)...
```

Now that the transversals and q's are stored, we verify if they are equal to each other. We check in Magma.

```
ff(A) eq q.1;
ff(B) eq q.2;
ff(C) eq q.3;
```

All three of them are true, so both q's and transversals are equal to each other. Now we proceed by writing a presentation for the central extension. To do so we have to check if the relations are equal to something else.

$$\langle a,b,c,d|a^{13}=?,b^4=?,c^2=?,b^{-2}a^{-1}b^2a^{-1}=?,b^{-1}a^{-3}ba^{-2}=?,aca^{-1}b^{-2}acb^2a=?,$$

$$aca^{-1}b^{-1}aca^{-1}b=?,bcb^{-2}cab^{-2}cb^2cb^{-1}c=?,ca^{-1}cb^{-1}ca^{-2}cacb^2ca=?,$$

$$ca^{-1}b^{-2}cbca^{-1}cb^2cacb^{-1}=?,d^2,(d,a),(d,b),(d,c)\rangle$$

We begin by checking if the order of the transversals and and q's are of order 13, 4, and 2. We have the order of A and $q1$ equal to 13, likewise for b and C. Thus, we have the order of a, b, and c match.

```
>Order(A), Order(q1);
13 13
>Order(B), Order(q2);
 4 4
>Order(C), Order(q3);
 2 2
```

Note, the order of A^{13} is identity, similar for the other two. This tells us that $a^{13} = b^4 = c^2 = Id$. So we proceed by checking the order of each relation but in this case if the relation is not identity we investigate and change the relation.

```
>Order(B^-2 * A^-1 * B^2 * A^-1),Order(q2^-2*q1^-1*q2^2*q1^-1);
True 1 1
>Order(B^-1*A^-3*B*A^-2), Order(q2^-1*q1^-3*q2*q1^-2);
True 1 1
>Order(A*C*A^-1*B^-2*A*C*B^2*A),
```

```
>Order(q1*q3*q1^-1*q2^-2*q1*q3*q2^2*q1);
True 1 1
>Order(A*C*A^-1*B^-1*A*C*A^-1*B),
>Order(q1*q3*q1^-1*q2^-1*q1*q3*q1^-1*q2);
True 1 1
>Order(B*C*B^-2*C*A*B^-2*C*B^2*C*B^-1*C),
>Order(q2*q3*q2^-2*q3*q1*q2^-2*q3*q2^2*q3*q2^-1*q3);
False 2 1
>Order(C*A^-1*C*B^-1*C*A^-2*C*A*C*B^2*C*A),
>Order(q3*q1^-1*q3*q2^-1*q3*q1^-2*q3*q1*q3*q2^2*q3*q1);
False 2 1
>Order(C*A^-1*B^-2*C*B*C*A^-1*C*B^2*C*A*C*B^-1),
>Order(q3*q1^-1*q2^-2*q3*q2*q3*q1^-1*q3*q2^2*q3*q1*q3*q2^-1);
False 2 1
```

For the first few relations we have $b^{-2}a^{-1}b^2a^{-1} = b^{-1}a^{-3}ba^{-2} = aca^{-1}b^{-2}acb^2a = aca^{-1}b^{-1}aca^{-1}b = Id$. Note, on the last three relations we have those three not equal to identity. In this situation we verify if they are equal to the center, d.

```
>B*C*B^-2*C*A*B^-2*C*B^2*C*B^-1*C eq D;
true
>C*A^-1*C*B^-1*C*A^-2*C*A*C*B^2*C*A eq D
true
>C*A^-1*B^-2*C*B*C*A^-1*C*B^2*C*A*C*B^-1 eq D;
true
```

Thus, the last relations are given as follow $bcb^{-2}cab^{-2}cb^2cb^{-1}c = d$, $ca^{-1}cb^{-1}ca^{-2}cacb^2ca = d$, and $ca^{-1}b^{-2}cbca^{-1}cb^2cacb^{-1} = d$. To finish proving the isomorphism type of the group we are going to add the new relations and the center to the presentation. We defined the new presentation as H. Note, since G has a center namley d we write each generator a, b, and c commute with d.

$$H < a,b,c,d >:= Group < a,b,c,d | a^{13}, b^4, c^2, b^{-2}a^{-1}b^2a^{-1}, b^{-1}a^{-3}ba^{-2}, aca^{-1}b^{-2}acb^2a,$$

$$aca^{-1}b^{-1}aca^{-1}b, bcb^{-2}cab^{-2}cb^2cb^{-1}c = d, ca^{-1}cb^{-1}ca^{-2}cacb^2ca = d,$$

$$ca^{-1}b^{-2}cbca^{-1}cb^2cacb^{-1} = d, d^2, (d,a), (d,b), (d,c) >;$$

Then, we verify if H is isomorphic to $G1$.

```
>#H;
```

```
58240
>#G1;
58240
>f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
>s:=IsIsomorphic(G1,H1);
>s;
true
```

True.

Therefore we have proven the isomorphism type of G, $2^{\bullet}S_z(8)$.

3.4 Semi-Direct Product

3.4.1 Extension of $13^2 : (4 \times 2)$

Isomorphism type of G

Since we know the definition of each extension we continue by proving the isomorphism type of a new finite homomorphic image using the following extensions direct and semi-direct product.

For the presentation listed below we discovered a finite homomorphic image, $G < x, y, t > := Group < x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^{-2},$

$t^2,$

$(t, y^x), ((y^{-1}x^{-1})t^{x^6})^4 >;$.

Now we find the isomorphism type of G of the following group.

Proof. The process in which we prove the isomorphism of G is:

First, we obtain the composition factor by using a program called MAGMA.

```
S:=Sym(13);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=S!(1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7);
N:=sub<S|aa,bb>;
```

```
> CompositionFactors(G1);
    G
```

```
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(13)
*
|  Cyclic(13)
1
```

The composition series of G:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5$ $(G_5 = 1)$

Leading us to the composition factors:

$$
\begin{aligned}
G &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5) \\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/1) \\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)G_4 \\
&= C_2 \cdot C_2 \cdot C_2 \cdot C_{13} \cdot C_{13}
\end{aligned}
$$

Now we check in MAGMA the Normal Lattice of G1.

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice

-----------------------

[13]  Order 1352  Length 1  Maximal Subgroups: 10 11 12
---
[12]  Order 676   Length 1  Maximal Subgroups: 9
[11]  Order 676   Length 1  Maximal Subgroups: 7 8 9
[10]  Order 676   Length 1  Maximal Subgroups: 9
---
[ 9]  Order 338   Length 1  Maximal Subgroups: 6
[ 8]  Order 338   Length 1  Maximal Subgroups: 4 6
[ 7]  Order 338   Length 1  Maximal Subgroups: 5 6
---
```

```
[ 6]  Order 169   Length 1  Maximal Subgroups: 2 3
[ 5]  Order 26    Length 1  Maximal Subgroups: 2
[ 4]  Order 26    Length 1  Maximal Subgroups: 3
---
[ 3]  Order 13    Length 1  Maximal Subgroups: 1
[ 2]  Order 13    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1     Length 1  Maximal Subgroups:
```

Now we find the maximum abelian group by applying the following loop in MAGMA.

```
> for i in [1..#NL] do
for> if IsAbelian(NL[i]) then i; end if; end for;
1
2
3
6
```

So we have the subgroup of [6] is the largest abelian group of $G1$. Refer back to the normal lattice, note that the order of the subgroup [6] is 169. Now we use $NL[6]$ to determine if the order of the subgroup[6] is either of the form 169 or 13×13. By MAGMA we determined the two different permutations, both permutations are of length 13. Now we are going stored both permutations in MAGMA as follow $A := G1!(2,4,3,10,7,9,17,18,6,11,15,5,13)$; and $B := G1!(1,8,19,25,16,$
$21,24,12,20,22,26,23,14)(2,5,11,18,9,10,4,13,15,6,17,7,3)$;. Now lets find G_3, G_3 is an extension of G_4 by G_3/G_4 which implies that $G_3/G_4 = C_{13}$ but $G_3 = C_{13} \cdot C_{13}$. To find what type of extension G_3 is we can check if it is a direct product using the following loop in MAGMA

```
> X:=AbelianGroup(GrpPerm,[13,13]);
> s:=IsIsomorphic(X,NL[6]);
> s;
true.
```

Up to this point we have proven that $G_3 = C_{13} \times C_{13}$.
Next, we find G_2 where G_2 is an extension of G_3 by G_2/G_3. Then $G_2 = (C_{13} \times C_{13}) \cdot C_2$. Now check on the Normal Lattice for the next order of G, in this case it can either be the subgroup $[7], [8]$, or [9]. Lets try $NL[7]$ with order 338 and check if Q is

a mixed extension. Is not a mixed estension since $IsAbelian(NL[7])$; is false. Is it a Central extension, false after checking in MAGMA $Center(G1)$;, and lastly is it a mixed extension, no since the order 2 is not listed on the normal lattice of $G1$. So our last option is a semi-direct product, so we have $G_2 = (C_{13} \times C_{13}) : 2$
Now to find G_1, G_1 is an extension of G_2 by G_1/G_2 where $G_1 = (C_{13} \times C_{13}) : 2 \cdot 2$ we check the Normal Lattice. The order of $NL[10]$; is of order 676.

```
> for g in NL[10] do
for> if Order(g) eq 4 and sub<NL[10]|g,NL[6]> eq NL[10]
for|if> then g; break;
for|if> end if; end for;
(1, 20, 12, 14)(2, 5, 6, 15)(3, 4, 17, 18)(8, 25, 24, 16)
(9, 13, 10, 11)(19, 23,21, 22)
```

The permutation listed above is stored and labeled as C in MAGMA.

```
 > C:=G1!(1, 20, 12, 14)(2, 5, 6, 15)(3, 4, 17, 18)
> (8, 25, 24, 16)(9, 13, 10, 11)(19, 23,21, 22);
```

Note C is of order 4, so our G_1 can be of the form $C_{13} \times C_{13} : 4$. We verify our assumption by checking the action C in A and B. We used the following loops in MAGMA.

```
> for i,j in [1..13] do
for>  if A^C eq A^i*B^j then i,j; end if; end for;
8 13
> for i,j in [1..13] do
for>  if B^C eq A^i*B^j then i,j; end if; end for;
13 8
```

Since we know $A^C = A^8$ and $B^C = B^8$, then we include them and write a new presentation.

```
> H<a,b,c>:=Group<a,b,c|a^13,b^13,(a,b),c^4,a^c=a^8,b^c=b^8>;
> #H;
676
```

To confirm $G_1 = C_{13} \times C_{13} : 4$ we check using the following loop.

```
> f,g1,k:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(g1,NL[10]);
> s;
true
```

True, we have proven G_1.

Now show what is $G_0 = (C_{13} \times C_{13}) : 4 \cdot 2$.

```
> for g in NL[13] do
for> if sub<NL[13]|g,NL[10]> eq NL[13]
for|if> then g; break; end if; end for;
(1, 2)(3, 8)(4, 12)(5, 14)(6, 16)(7, 19)
(9, 22)(10, 20)(11, 23)(13, 24)
(15,21)(17, 25)(18, 26)
```

Since the of order of the permutation is 2 we have to check if it is a central extension.

Center(G1); is false since the order is 1.

Check if it is a mixed extension by using IsAbelian(NL[10]);, false. Our only options left to be checked are direct and semi-direct product. If we the action of A^B; we get equal to A. That implies semi-direct is eliminated leaving us with direct product.

The element given above we are going to stored and labeled it as D, D:=G1!(1, 2)(3, 8)(4, 12)(5, 14)(6, 16)(7, 19)(9, 22)(10, 20)(11, 23)(13, 24)(15,21)(17, 25)(18, 26);

To find the action of D in A and B we are going to use the SchreierSystem, the SchreierSystem is used to convert permutations into words or finding the actions of other elements. So we check $A^D = ?, B^D = ?$, and $C^D = ?$.

```
 >  NL13:=sub<G1|A,B,C>;
>  NL13 eq NL[10];
true
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
>  ArrayP:=[Id(N): i in [1..1352]];
>  for i in [2..1352] do
for>  P:=[Id(N): l in [1..#Sch[i]]];
for>  for j in [1..#Sch[i]] do
for|for>  if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for>  if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
for|for>  if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for>  if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
for|for>  if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
for|for>  if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
for|for>  end for;
for>  PP:=Id(N);
for>  for k in [1..#P] do
for|for>  PP:=PP*P[k]; end for;
for>  ArrayP[i]:=PP;
for>  end for;
```

```
>  for i in [1..1352] do if ArrayP[i] eq A^D then Sch[i];
end if; end for;
a * c * b^-1 * c^-1 * b^-1
>  for i in [1..1352] do if ArrayP[i] eq B^D then Sch[i];
end if; end for;
b^-1
>  for i in [1..1352] do if ArrayP[i] eq C^D then Sch[i];
end if; end for;
b^-1 * c * b^-1
```

Since, $A^D = acb^{-1}c^{-1}b^{-1}, B^D = b^{-1}$, and $C^D = b^{-1}cb^{-1}$ now we add them the a new presentation.

```
> H<a,b,c,d>:=Group<a,b,c,d|a^13,b^13,(a,b), c^4, a^c=a^8,
> b^c=b^8, d^2, a^d=a * c * b^-1 * c^-1 * b^-1,b^d=b^-1,
>c^d=b^-1 * c * b^-1>;
> #H;
1352
> #G1;
1352
> f,g2,k:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(g2,G1);
> s;
true
```

True, we have proven the isomorphism type of $G \cong 13^2 : (4 \times 2)$.

3.4.2 Direct Product Extension

To find the isomorphism type of a control group,N of 2^{*12} we use the previous method to prove that $N \cong S_4 \times 2$. We start by checking the composition factor of the control group, N.

Isomorphism type of N

To complete the progenitor $2^{*12} : N$ we need to prove that $N \cong S_4 \times 2$. Given the following generators and presentation of N we run the following loops to find the composition factor and normal lattice.

```
>S:=Sym(12);
>vv:=S!(1, 2)(3, 6)(4, 8)(5, 9)(7, 11);
```

```
>ww:=S!(1, 3, 11)(2, 7, 6)(4, 8, 12)(5, 9, 10);
>xx:=S!(2, 8)(3, 9)(7, 10)(11, 12);
>yy:=S!(1, 4)(5, 6)(7, 12)(10, 11);
>zz:=S!(1, 5)(2, 9)(3, 8)(4, 6);
>N:=sub<S|vv,ww,xx,yy,zz>;
>#N;
48
```

Then the extension is found under the composition factor of the control group N. We begin with the prove.

```
>CompositionFactors(N);
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

Now print out the normalLattice of N.

```
>NL:=NormalLattice(N);
>NL;
Normal subgroup lattice
-----------------------

[9]  Order 48  Length 1  Maximal Subgroups: 6 7 8
---
[8]  Order 24  Length 1  Maximal Subgroups: 5
[7]  Order 24  Length 1  Maximal Subgroups: 5
[6]  Order 24  Length 1  Maximal Subgroups: 4 5
---
[5]  Order 12  Length 1  Maximal Subgroups: 3
[4]  Order 8   Length 1  Maximal Subgroups: 2 3
---
[3]  Order 4   Length 1  Maximal Subgroups: 1
---
[2]  Order 2   Length 1  Maximal Subgroups: 1
```

```
---
[1]  Order 1   Length 1  Maximal Subgroups:
```

To find the extension of the control group we begin by checking if the isomorphism type has a center, $Center(N)$. The center of N is of order two. But, before we state that the extension of N has a center of order two, we find the largest abelian group. The largest abelian group is the normal subgroup [4] where $NL[4]$ is of order 8. Note, the order of the largest abelian group is larger than the center, so the extension of N does not have a center. Note that the order of the normal subgroup of NL[8] is equal to 24. It is clear that NL[8] is congruent to $Sym(4)$, since the order of $Sym(4)$ is also equal to 24. So we check our assumptions.

```
>IsIsomorphic(NL[8],Sym(4));
true
```

Now we find and store the generators of $NL[8]$ as X, Y, V, and W:

```
>Generators(NL[8]);
 (1, 9)(2, 5)(3, 4)(6, 8)(7, 11),
 (1, 11, 3)(2, 6, 7)(4, 12, 8)(5, 10, 9),
 (1, 6)(2, 9)(3, 8)(4, 5)(7, 12)(10, 11),
 (1, 4)(2, 8)(3, 9)(5, 6)(7, 11)(10, 12)
>X:=N!(1, 9)(2, 5)(3, 4)(6, 8)(7, 11);
>Y:=N!    (1, 11, 3)(2, 6, 7)(4, 12, 8)(5, 10, 9);
>V:=N!(1, 6)(2, 9)(3, 8)(4, 5)(7, 12)(10, 11);
>W:=N!(1, 4)(2, 8)(3, 9)(5, 6)(7, 11)(10, 12);
```

We continue by factoring N by $NL[8]$.

```
>q,ff:=quo<N|NL[8]>;
>q;
```

Since q is C_2 we check if the extension of S_4 and C_2 has a direct product.

```
>IsIsomorphic(N,DirectProduct(Sym(4),CyclicGroup(2)));
true
```

Then we write a presentation of $Sym(4)$.

```
>FPGroup(NL[8]);
x^2, y^3,v^2,w^2,y^-1 * v * y * w,
(x * w)^2,(vw * w)^2,y^-1 * x * y^-1 * x * v
```

Check if the presentation is isomorphic to $Sym(4)$

```
>H<x,y,v,w,z>:=Group<x,y,v,w,z|x^2,y^3,
>v^2,w^2,y^-1 * v * y * w,(x * w)^2,
>(v* w)^2,y^-1 * x * y^-1 * x * v>;
>#H;
24
>f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
>s:=IsIsomorphic(H1,Sym(4));
>s;
true
```

Since we have the presentation of $Sym(4)$ and is a normal subgroup. Then, by definition the extension of Sym(4) and q is a direct product. Now we label q as Z and we write the generator z commute with x, y, v, and w by definition. Thus, $(z, x), (z, y), (z, v)$, and (z, w) is included to the presentation of H.

```
>H<x,y,v,w,z>:=Group<x,y,v,w,z|x^2,y^3,v^2,
>w^2,y^-1 * v * y * w,(x * w)^2,(v* w)^2,
>y^-1 * x * y^-1 * x * v,z^2,(z,x),(z,y),(z,v),(z,w)>;
>#H;
48
>f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
>s:=IsIsomorphic(H2,N);
>s;
true
```

True, N is isomorphic to $S_4 \times 2$. Therefore, we have completed the progenitor 2^{*12} : $S_4 \times 2$.

3.5 Mixed Extension

3.5.1 Mixed Extension: $2^4 :^{\bullet} S_5$

Consider the group $G =$

$$\langle x, y, t | x^2, y^6, (yxy^{-1}x)^2, (xy^{-1})^5, (t, x^y),$$

$$t^2,$$

$$(y^2xy^{-2})^4, (yxyt)^2, (yxt)^5\rangle$$

The composition factors of G.

```
CompositionFactors(G1);
   G
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

 NL:=NormalLattice(G1);
 NL;

Normal subgroup lattice
-----------------------

[4]  Order 1920  Length 1  Maximal Subgroups: 3
---
[3]  Order 960   Length 1  Maximal Subgroups: 2
---
[2]  Order 16    Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

We check if there is a center to see if a central extension is possible. In this case the order of the center is 1, so we rule out a central extension. Note, there is no subgroup with order 2, then direct product is also ruled out. The last two options we have are semi-direct product or mixed extension. To find the extension of the group we check the largest abelian group of G, which is $NL[2]$ of order 16. So that means K is the largest abelian group of order 16. K is either in the form $16, 2 \times 2 \times 2 \times 2$, or 2^4. To verified our assumptions we check by using Magma.

```
X:=[2,2,2,2];
 IsIsomorphic(NL[2],AbelianGroup(GrpPerm,(X)));
 true
```

Hence, $K \cong 2^4$. $FPGroup(NL[2])$ gives a presentation of $K = 2^4$.

$$H < a,b,c,d >:= Group < a,b,c,d|a^2, b^2, c^2, d^2, (ab)^2, (ac)^2, (ad)^2, (bc)^2, (bd)^2, (dc)^2 >;$$

$$\#H = 16$$

Therefore, the presentation of 2^4 is correct.

Now we factor G by K and use the quotient group Q. Now factor G by K using the loop in Magma.

```
>q,ff:=quo<G1|NL[2]>;
>q;
>#q;
120
> CompositionFactors(q);
    G
    |  Cyclic(2)
    *
    |  Alternating(5)
    1
>nl:=NormalLattice(q); nl;
Normal subgroup lattice
-----------------------

[3]  Order 120  Length 1  Maximal Subgroups: 2
---
[2]  Order 60   Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

Since we have Q we also check if it has a center. False, the order of the center is 1 so we do not have a central extension. Is it a direct product, to be a direct-product we need the normal subgroups of order 60 and 2 be found on the normal lattice. In this case the orders of 2 and 60 is not in the normal lattice, so a direct product is ruled out. That leaves us with two options a semi-direct or mixed extension. If we have a semi-direct product where 60 semi-direct 2, then it is possible to have S_5 of order 120.

```
>IsIsomorphic(q,SymmetricGroup(5));
true
```

Therefore, $Q \cong S_5$.

We write a presentation for S_5

$$H < e, f, g >:= Group < e, f, g|e^2, f^6, g^2, (fge)^2, egef^{-1}gf^{-1}g, ef^{-3}ef^{-1}ef^{-1}g >;$$

$$\#H = 120$$

Hence, the presentation for S_5 is correct.

Up to this point we have proven the presentations of $K \cong 2^4$ and $Q \cong S_5$. Now we need to find if G is semi-direct or mixed extension. To show if $G \cong 2^4 :^{\bullet} S_5$ we need to complete the folowing presentation.

$$\langle a, b, c, d, e, f, g|a^2, b^2, c^2, d^2, (ab)^2, (ac)^2, (ad)^2, (bc)^2, (bd)^2, (dc)^2,$$

$$e^2, f^6, g^2, (fge)^2 =?, egef^{-1}gf^{-1}g =?, ef^{-3}ef^{-1}ef^{-1}g =?,$$

$$a^e =?, a^f =?, a^g =?, b^e =?, b^f =?, b^g =?, c^e =?, c^f =?, c^g =?, d^e =?, d^f =?, d^g =?\rangle$$

To complete the presentation we convert the generators of S_5 in terms of the generators of 2^4.

```
T:=Transversal(G1,NL[2]);
```

Then we check if the transversals of NL[2] and the generators of q match.

```
>ff(T[2]) eq q.1;
true
 >ff(T[3]) eq q.2;
true
 >ff(T[4]) eq q.3;
true
```

We have proven that the transversals and the generators of q match.

Now we find the generators of $NL[2]$ and stored them.

```
Generators(NL[2]);
A:=G1!(1, 2)(3, 13)(4, 9)(5, 8)(6, 7)(10, 12)(11, 16)(14, 15);
 B:=G1!(1, 5)(2, 8)(3, 14)(4, 12)(6, 11)(7, 16)(9, 10)(13, 15);
 C:=G1!(1, 3)(2, 13)(4, 11)(5, 14)(6, 12)(7, 10)(8, 15)(9, 16);
 D:=G1!(1, 4)(2, 9)(3, 11)(5, 12)(6, 14)(7, 15)(8, 10)(13, 16);
 NL[2] eq sub<G1|A,B,C,D>;
true
```

Note, each generator of $NL[2]$ is stored as A, B, C, and D. We also stored the transversal as E, F, and G. Now we find the action of T on $NL[2]$

```
>T[2];
(2, 3)(6, 10)(8, 14)(9, 11)
>E:=G1!(2, 3)(6, 10)(8, 14)(9, 11);
>T[3];
(2, 4)(3, 5, 7)(6, 11, 8, 15, 13, 12)(10, 14, 16)
>F:=G1!(2, 4)(3, 5, 7)(6, 11, 8, 15, 13, 12)(10, 14, 16);
>T[4];
(1, 2)(3, 6)(4, 8)(5, 9)(7, 13)(10, 12)*/
>G:=G1!(1, 2)(3, 6)(4, 8)(5, 9)(7, 13)(10, 12);
```

Now we check and verify the presentation of q in terms of the transversals.

$$e^2, f^6, g^2, (fge)^2 =?, egef^{-1}gf^{-1}g =?, ef^{-3}ef^{-1}ef^{-1}g =?,$$

```
T[2] = .1^2 = e
T[3] = .2^6 = f
T[4] = .3^2 = g

 >ff(T[2]) eq q.1;
 true
 >T[2]^2;
Id(G1)
 >T[4]^2;
Id(G1)
 >T[3]^6;
Id(G1)
```

Up to this point it is true, so we continue.

```
 >Order((T[3]*T[4]*T[2])^2);
 16
```

Note, the order is 16 not 2. So we find the action of T on $NL[2]$.

```
for i,j,k,l in [1..2] do
if (T[3]*T[4]*T[2])^2 eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
/*2 2 1 1*/
```

Thus, $(fge)^2 = CD$

We apply the same process to complete the presentation.

```
> Order(T[2]*T[4]*T[2]*T[3]^-1*T[4]*T[3]^-1*T[4]);
2

for i,j,k,l in [1..2] do
if T[2]*T[4]*T[2]*T[3]^-1*T[4]*T[3]^-1*T[4] eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 1 1 2
```

Then, $egef^{-1}gf^{-1}g = ABC$

```
Order(T[2]*T[3]^-3*T[2]*T[3]^-1*T[2]*T[3]^-1*T[4]);
/*2*/
for i,j,k,l in [1..2] do
if T[2]*T[3]^-3*T[2]*T[3]^-1*T[2]*T[3]^-1*T[4] eq
A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
 1 2 2 2
```

So, $ef^{-3}ef^{-1}g = A$.

Therefore, $(fge)^2 = cd, egef^{-1}gf^{-1}g = abc, ef^{-3}ef^{-1}ef^{-1}g = a.$

Now we find the action of A^E =?, A^F =?, A^G =?, B^E =?, B^F =?, B^G =? ...D^G =? where the following loop is used in Magma.

```
>for i,j,k,l in [1..2] do if A^E eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 2 1 2
>for i,j,k,l in [1..2] do if A^F eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
2 2 2 1
>for i,j,k,l in [1..2] do if A^G eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
1 2 2 2
>for i,j,k,l in [1..2] do if B^E eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 1 2 2
>for i,j,k,l in [1..2] do if B^F eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
```

```
1 1 1 1
>for i,j,k,l in [1..2] do if B^G eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
2 2 2 1
>for i,j,k,l in [1..2] do if C^E eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 2 2 2
>for i,j,k,l in [1..2] do if C^F eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 1 2 2
>for i,j,k,l in [1..2] do if C^G eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 1 1 1
>for i,j,k,l in [1..2] do if D^E eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
2 2 2 1
>for i,j,k,l in [1..2] do if D^F eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 2 2 2
>for i,j,k,l in [1..2] do if D^G eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 1 2 2
```

To get the order of G we are going to combine the presentations. Thus, the isomorphism type of G is congruent to $2^4 :^{\bullet} S_5$.

$$H < a,b,c,d,e,f,g >:= Group < a,b,c,d,e,f,g | a^2, b^2, c^2, d^2, (ab)^2, (ac)^2, (ad)^2, (bc)^2,$$

$$(bd)^2, (dc)^2, e^2, f^6, g^2, (fge)^2 = cd, (egef^{-1}gf^{-1}g) = abc, (ef^{-3}ef^{-1}ef^{-1}g) = a, a^e = c,$$

$$a^f = d, a^g = a, b^e = b, b^f = abcd, b^g = d, c^e = a, c^f = b, c^g = abcd, d^e = d, d^f = a, d^g = b$$

$$\#H; = 1920$$

```
> f,g,k:=CosetAction(H,sub<H|Id(H)>);
>s:=IsIsomorphic(G1,g);
>s;
 true
```

Therefore, $G \cong 2^4 :^{\bullet} S_5$.

Chapter 4

Double Coset Enumeration

4.1 Factor $8 \times PSL_2(7)$ by the Order 8

To have a better image of the group G we are going to get rid of the $Cyclics(2)$ from the Composition Factors of the group G. We have the following, $G \cong$

$$\langle x, y, z, w, t | x^2, y^3, z^2, w^2, (y^{-1}x)^2, y^{-1}zyw, (xz)^2, (zw)^2, yzy^{-1}zw,$$

$$, t^2,$$

$$(t, xy), (xy^{-1}wt)^3, (yt)^3 \rangle.$$

Now we compute the $CompositionFactors$ and the $NormalLattice$ of G, listed below

```
CompositionFactors(G1);
   G
    |  A(1, 7)                      = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1.

 NL:=NormalLattice(G1);
 NL;
```

```
Normal subgroup lattice
-----------------------

[3]  Order 1344  Length 1  Maximal Subgroups: 2
---
[2]  Order 8     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

Factor $8 \times PSL_2(7)$ by the order 8.

```
IsAbelian(NL[2]);
X:=AbelianGroup(GrpPerm,[2,2,2]);
IsIsomorphic(NL[2],X);
 NL[2];
```

Since we get three different permutations we stored each permutation as A, B, and C and we convert the permutations into words using the SchreierSystem.

```
 N:=G1;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(t); end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..1344] do if ArrayP[i] eq A
then Sch[i]; end if; end for;
for i in [1..1344] do if ArrayP[i] eq B
then Sch[i]; end if; end for;
for i in [1..1344] do if ArrayP[i] eq C
```

```
then Sch[i]; end if; end for;
```

Now that we have converted each permutations into words, we factor the group G by $(wt)^3$, $xwytzwt$, and $tztxwytwt$ and we get the following:

$$\langle x,y,z,w,t|x^2,y^3,z^2,w^2,(y^{-1}x)^2,y^{-1}zyw,(xz)^2,(zw)^2,yzy^{-1}zw,$$

$$,t^2,$$

$$(t,xy),(xy^{-1}wt)^3,(yt)^3,(wt)^3,xwytzwt,tztxwytwt\rangle.$$

Now we check the composition factor of G.

```
CompositionFactors(G1);
   G
    |  A(1, 7)                    = L(2, 7)
    1.
```

Therefore, $G \cong PSL(2,7)$.

4.2 Manual Construction of $PSL_2(7)$ Over S_4

4.2.1 Manual Double Coset Enumeration of $PSL_2(7)$

To obtain a homomorphic image we factor the progenitor $2^{*12}:S_4$ by the following relations $(yt)^3$, $(wt)^3$, $xwytzwt$, and $tztxwytwt$. We denote G where $G \cong 2^{*12}:S_4$ be a symmetric presentation given by:

$$\langle x,y,z,w,t|x^2,y^3,z^2,w^2,(y^{-1}x)^2,y^{-1}zyw,(xz)^2,(zw)^2,yz*y^{-1}zw,$$

$$t^2,$$

$$(t,xy),(xy^{-1}wt)^3,(yt)^3,(wt)^3,xwytzwt,tztxwytwt\rangle$$

where the control group $N \cong S_4 =$

$$\langle x,y,z,w,|x^2,y^3,z^2,w^2,(y^{-1}x)^2,y^{-1}zyw,(xz)^2,(zw)^2,yz*y^{-1}zw\rangle.$$

So we have the following:

$$G \cong \frac{2^{*12} : S_4}{(yt)^3, (wt)^3, zxwywt_{12}t_6 = t_1t_9, xwyzwt_{12} = t_1} \cong PSL(2,7).$$

Let $x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$, $y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$,
$z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$, and $w \sim (1,9)(2,7)(3,8)(4,10)(5,11)(6,12)$
where $t \sim t_1$.

The first relation, $(yt)^3$, is expand as follow:

$$\begin{aligned}
(yt)^3 &= 1\\
ytytyt &= 1\\
ytyyy^{-1}tyt &= 1\\
yty^2t^yt &= 1\\
y^3t^{(y^2)}t^yt &= 1\\
y^3t^{(y^2)}t^yt &= 1\\
t_4t_7t_1 &= 1
\end{aligned}$$

Conjugate $t_4t_7t_1 = 1$ by the elements of N, this will give us more relations. Before we conjugate change $t_4t_7t_1$ to $[4,7,1]$.

$[4,7,1]^N = [4,7,1], [1,7,4], [1,4,7], [3,11,6], [10,2,9], [7,4,1], [6,11,3], [9,2,10], [4,1,7], [7,1,4], [6,3,11], [9,10,2,][8,5,12], [12,5,8], [12,8,5], [11,3,6], [2,10,9], [5,8,12], [3,6,11], [10,9,2], [11,6,3], [2,9,10], [8,12,5], [5,12,8]$

Expand second relation $(wt)^3$:

$$\begin{aligned}
(wt)^3 &= 1\\
wtwtwt &= 1\\
wtwww^{-1}twt &= 1\\
wtw^2t^wt &= 1\\
w^3t^{(w^2)}t^wt &= 1\\
w^3t^{(w^2)}t^wt &= 1\\
(1,9)(2,7)(3,8)(4,10)(5,11)(6,12)t_1t_9t_1 &= 1
\end{aligned}$$

Now change $t_1 t_9 t_1$ to $[1, 9, 1]$ and conjugate by the elements of N.

$((1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[1, 9, 1])^N$

$= (1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[7, 5, 7]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[6, 12, 6]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[9, 1, 9]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[4, 3, 4]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[11, 2, 11]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[2, 11, 2]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[12, 6, 12]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[8, 10, 8]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[5, 7, 5]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[3, 4, 3]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[10, 8, 10]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[4, 8, 4]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[1, 6, 1]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[3, 10, 3]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[10, 3, 10]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[7, 2, 7]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[6, 1, 6]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[9, 12, 9]$,

$(1, 12)(2, 11)(3, 10)(4, 8)(5, 7)(6, 9)[8, 4, 8]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[5, 11, 5]$,

$(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)[12, 9, 12]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[11, 5, 11]$,

$(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12)[2, 7, 2]$

Expand third relation $xwytzwt$:

$$\begin{aligned} xwytzwt &= 1 \\ xwy(zw)(zw)^{-1}t(zw)t &= 1 \\ xwy(zw)t^{zw}t &= 1 \\ xwy(zw)t_{12}t_1 &= 1 \end{aligned}$$

Now change $t_{12}t_1$ to $[12,1]$ and conjugate by the elements of N.

$((2,3)(4,7)(5,8)(6,9)(10,11)[12,1])^N = (2,3)(4,7)(5,8)(6,9)(10,11)[12,1]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[11,7]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[9,6]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[6,9]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[10,4]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[7,11]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[5,2]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[1,12]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[3,8]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[2,5]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[8,3]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[4,10]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[10,4]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[12,1]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[8,3]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[4,10]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[11,7]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[9,6]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[6,9]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[3,8]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[2,5]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[1,12]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[7,11]$, and

$(1,3)(4,6)(7,11)(8,12)(9,10)[5,2]$

For the last relation $tztxwytwt$ we have:

$$\begin{aligned} tztxwytwt &= 1 \\ tztxwyww^{-1}twt &= 1 \\ tztxwywt^{w}t &= 1 \\ tzxwywt^{(xwyw)}t^{w}t &= 1 \\ zxwywt^{(zxwyw)}t^{(xwyw)}t^{w}t &= 1 \\ zxwywt_{12}t_6t_9t_1 &= 1 \end{aligned}$$

Change $t_{12}t_6t_9t_1$ to $[12,6,9,1]$ and conjugate by the elements of N

$((1,12)(2,10)(3,11)(4,5)(7,8)[12,6,9,1])^N = (1,12)(2,10)(3,11)(4,5)(7,8)[12,6,9,1]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[11,2,5,7]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[9,1,12,6]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[6,12,1,9]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[10,8,3,4]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[7,5,2,11]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[5,7,11,2]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[1,9,6,12]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[3,4,10,8]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[2,11,7,5]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[8,10,4,3]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[4,3,8,10]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[10,3,8,4]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[12,9,6,1]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[8,4,10,3]$,

$(1,2)(4,10)(5,12)(6,11)(7,9)[4,8,3,10]$,

$(1,3)(4,6)(7,11)(8,12)(9,10)[11,5,2,7]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[9,12,1,6]$,

$(2,3)(4,7)(5,8)(6,9)(10,11)[6,1,12,9]$,

$(1,7)(2,9)(3,8)(5,6)(11,12)[3,10,4,8]$,

$(1,4)(2,5)(3,6)(8,9)(10,12)[2,7,11,5]$,

$(1,12)(2,10)(3,11)(4,5)(7,8)[1,6,9,12]$,
$(1,3)(4,6)(7,11)(8,12)(9,10)[7,2,5,11]$, and
$(1,4)(2,5)(3,6)(8,9)(10,12)[5,11,7,2]$.

Now that we have obtain the homomorphic image we can start constructing the a manual double coset enumeration of G over N. Let w be a word in $t_i s$ and $[w]$ be the double coset, NwN.

NeN

The process on constructing a manual double coset we first begin by defining a double coset which is $NeN = \{Ne^n | n \epsilon N\}$. So on the first double coset we have $NeN = \{Ne^n | n \epsilon N\}$ where we denoted by $[*]$. In the double coset $[*]$ there is only one single coset, namely N. The coset stabiliser of N is N and is transitive on twelve letters namely $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}\}$.Thus, has a single orbit {1,2,3,4,5,6,7,8,9,10,11,12}. Take a representative from the single orbit say 1 and do right multiplication to Ne and we check which double coset Nt_1 belongs. In this case $Net_1 = Nt_1N$ is a new double coset denoted as [1]. Since we have 12 elements on the orbit then 12 things go to the new double coset, [1].

Nt_1N

Continuing with the new double coset Nt_1N. If we conjugate Nt_1 by element of N we get $Nt_1N = Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}$. Now we find the point stabiliser N^1 and coset stabiliser $N^{(1)}$ to determine the amount of single cosets that are in the new double coset [1]. Note the coset stabiliser $N^{(1)}$ is equal to the point stabiliser N^1. Since $t_1 = xyt_{12}$ where $Nt_{12} \, \epsilon \, Nt_1$ then $(2,3)(4,7)(5,8)(6,9)(10,11)$ ϵ N. Hence, $(2,3)(4,7)(5,8)(6,9)(10,11) \, \epsilon \, N^{(1)}$. Therefore:

$$N^{(1)} \geq \langle (2,3)(4,7)(5,8)(6,9)(10,11) \rangle$$

Then the order of the coset stabiliser of $N^{(1)}$ denoted as $|N^{(1)}| = 2$. So the number of single cosets in $N^{(1)}$ is $\frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 6$. The single coset representatives of Nt_1N are N, Nt_{12}, Nt_{10}, Nt_{11}, Nt_9, Nt_8, and Nt_5. Note, we have the following equal names on the double coset [1] which are $1 \sim 12$, $4 \sim 10$, $7 \sim 11$, $6 \sim 9$, $9 \sim 6$, $3 \sim 8$, $10 \sim 4$, $11 \sim 7$, $2 \sim 5$, $12 \sim 1$, $8 \sim 3$, and $5 \sim 2$. Since we know there are six single cosets in the new double coset [1] now we find the orbits of $N^{(1)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12\}$. The orbits are:

$$\mathbb{O} = \{1\}, \{12\}, \{2,3\}, \{4,7\}, \{5,8\}, \{6,9\}, \{10,11\}.$$

Now take a representative $t_1, t_{12}, t_2, t_4, t_5, t_6$, and t_{10} respectively and right multiply to the coset representative Nt_1. So we get the following single cosets representatives Nt_1t_1, Nt_1t_{12}, Nt_1t_2, Nt_1t_4, Nt_1t_5, and Nt_1t_6. Now check if there are new double coset or if they equal to other existing double cosets. Since we have 2^{*12} where all $t's$ are of order two then we have $Nt_1t_1 = N \in [*]$. Note, the orbit with the element 1 has only one element, then one symmetric generator goes back to the double coset $[*]$. For the single cosets $Nt_1t_{12}, Nt_1t_2, Nt_1t_4, Nt_1t_5$, and Nt_1t_6 requires further investigation to determine if they are new double cosets or if they go back to other existing double cosets. To verify where the rest of the single cosets go, we need to consider the relations $t_1(2,3)(4,7)(5,8)(6,9)(10,11)t_{12}$, $t_4t_7t_1 = e$, $(1,9)(2,7)(3,8)(4,10)(5,11)(6,12)t_1t_9 = t_1$ and $(1,12)(2,10)(3,11)(4,5)(7,8)t_{12}t_6 = t_1t_9$. Now verify where Nt_1t_{12} belongs.
We have $\underline{t_1}t_{12} = (2,3)(4,7)(5,8)(6,9)(10,11)t_{12}t_{12} = N$. This implies $Nt_1t_{12} = N \in [*]$. Since there is only one element, 12, in the orbit, one symmetric generator goes to the double coset $[*]$. Using the relation $t_4t_7t_1 = e$ we have $t_1t_4 = t_7$ this implies $Nt_1t_4 = Nt_7 \in [1]$ so two symmetric generators will loop back to the double coset [1] since there are two elements in the orbit $\{4,7\}$.
Then for Nt_1t_5 we get:

$$15 = 1\underline{5}$$

$$= 18\underline{12}$$

$$= 18(1,12)(2,10)(3,11)(4,5)(7,8)196$$

$$(1,12)(2,10)(3,11)(4,5)(7,8)12\underline{71}96$$

$$= (1,12)(2,10)(3,11)(4,5)(7,8)\underline{12}496$$

$$= (1,12)(2,11)(3,10)(4,8)(5,7)(6,9)\underline{14}96$$

$$= (1,12)(2,11)(3,10)(4,8)(5,7)(6,9)7\underline{96}$$

$$= (1,12)(2,3)(4,7)(5,8)(6,9)(10,11)8$$

$$Nt_1t_5 = Nt_8 \in [1]$$

Hence two generators will loop back to the double coset [1].

Now for $= Nt_1t_2$ we check and verify.

$$12 = \underline{1}2$$

$$= (1,9)(2,7)...19\underline{2}$$

$$= (1,3)(4,6)310\underline{5}$$

$$= (1,3)(4,6)3108\underline{12}$$

$$= (1,12)(2,10)(3,11)...\underline{112}7169$$

$$= (1,3)(4,6)(7,11)...\underline{757}169$$

$$= (1,12)(2,11)(3,10)(4,8)1\underline{69}$$

$$= (1,12)(2,10)(3,11)12$$

$$Nt_1t_2 = Nt_{12} \in [1].$$

Hence, Nt_1t_2 is in the double coset [1] so two symmetric generators will loop back to [1].

For Nt_1t_6 we check and verify where it belongs.

$$16 = 1\underline{6}$$

$$= 1(2,3)(4,7)(5,8)(6,9)(10,11)9112$$

$$= (2,3)(4,7)(5,8)(6,9)(10,11)\underline{19}112$$

$$= (2,3)(4,7)(5,8)(6,9)(10,11)(1,12)(2,10)(3,11)(4,5)(7,8)12611\underline{2}$$

$$(1,12)(2,11)(3,10)(4,8)(5,7)(6,9)(2,3)(4,7)(5,8)(6,9)(10,11)41$$

$$(1,12)(2,11)(3,10)(4,8)(5,7)(6,9)12(2,3)(4,7)(5,8)(6,9)(10,11)7$$

$$Nt_1t_6 = Nt_7 \in [1]$$

. So two symmetric generators will loop back to the double coset [1].
Following this procedure for the next single coset Nt_1t_{10} we get:

$$t_1t_{10} = 1\underline{10}$$

$$= 1(1,7)(2,9)(3,8)(5,6)(11,12)4$$

$$= (1,7)(2,9)(3,8)(5,6)(11,12)\underline{74}$$

$$= (1,7)(2,9)(3,8)(5,6)(11,12)1$$

$$Nt_1t_{10} = Nt_1 \in [1].$$

Since there are two elements in the orbit $\{10,11\}$ then there are two symmetric generators that will loop back to the double coset [1].

We have completed the double coset enumeration of G, since the set of right cosets is closed under right multiplication. Thus the index of N in G is 7. We have concluded the following:

$$G = NeN \cup Nt_1N$$

where

$$G = \frac{2^{*12} : S_4}{(yt)^3, (wt)^3, zxwywt_{12}t_6 = t_1t_9, xwyzwt_{12} = t_1}$$

and

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} \times |N|$$

$|G| \leq (1+6) \times 24$
$|G| \leq 168.$
The Cayley graph summarizes the information listed above.

Figure 4.1: Cayley graph of $PSL(2,7)$ over S_4

4.2.2 Permutation Representation of $PSL_2(7)$ over S_4

Table 4.1: Permutation Representation of $PSL_2(13)$ over S_4

Cosets	$x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$	$y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$	$z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$	$w \sim 1,9)(2,7)(3,8)(4,10)(5,11)(6,12)$	$t \sim t_1$
1. N	1. N	1. N	1. N	1. N	2. Nt_1
2. Nt_{12}	3. Nt_{10}	4. Nt_{11}	3. Nt_{11}	5.$Nt_6 = Nt_9$	1. $Nt_{12}t_1 = N$
3. Nt_{10}	2. Nt_{12}	2. Nt_{12}	2. Nt_7	3.$Nt_4 = Nt_{10}$	4. $Nt_{10}t_1 = Nt_4t_1 = Nt_{11}$
4. Nt_{11}	4. Nt_{11}	3. Nt_{10}	6. Nt_{12}	7.Nt_5	3. $Nt_{11}t_1 = Nt_7t_1 = Nt_{10}$
5. Nt_9	6. Nt_8	7. Nt_5	9. Nt_8t_3	2.$Nt_1 = Nt_{12}$	5. $Nt_9t_1 = Nt_9$
6. Nt_8	5. Nt_9	5. $Nt_6 = Nt_9$	4. Nt_{10}	6.$Nt_3 = Nt_8$	6. $Nt_8t_1 = Nt_8$
7. Nt_5	7. $Nt_2 = Nt_5$	6. $Nt_3 = Nt_6$	10. Nt_7t_{11}	4.Nt_{11}	7. $Nt_5t_1 = Nt_5$

We get the following permutations:

$\phi(x) = (2,3)(5,6)$,

$\phi(y) = (2,4,3)(5,7,6)$,

$\phi(z) = (2,5)(3,6)$,

$\phi(w) = (2,5)(4,7)$, and

$\phi(t) = (1,2)(3,4)$.

Hence, we have $\phi : 2^{*12} : S_4 \to S_7$ since G acts on $X = N, Nt_{12}, Nt_{10}, Nt_{11}, Nt_9, Nt_8, Nt_5$ with $|X| = 7$. Then $\phi(G) = \langle \phi(x), \phi(y), \phi(z), \phi(w), \phi(t) \rangle$. To prove that $\phi(G) = \langle \phi(x), \phi(y), \phi(z), \phi(w), \phi(t) \rangle$ is a homomorphic image $G = 2^{*12} : S_4$, the following conditions must be satisfied:

1. $\phi(N) \cong S_4$

2. $\phi(t)$has twelve conjugates under conjugation by $\phi(N)$

3. $\phi(N)$ acts as S_4 on the twelve conjugates of $\phi(t)$ by conjugates

Proof. Let $\phi(N) = \langle \phi(x), \phi(y), \phi(z), \phi(w) \rangle$.

1. We want to show $\phi(N) = \langle \phi(x), \phi(y), \phi(z), \phi(w) \rangle \cong S_4$.
   Let the presentation of S_4 be

$$x^2, y^3, z^2, w^2, (y^{-1}x)^2 = 1, y^{-1}zyw = 1, (xz)^2, (zw)^2, yzy^{-1}zw = 1.$$

   Now we check if the presentation holds for $\phi(N)$.
   $|\phi(x)| = |(2,3)(5,6)| = 2, |\phi(y)| = |(2,4,3)(5,7,6)| = 3$, $|\phi(z)| = (2,5)(3,6) = 2$,
   $|\phi(w)| = (2,5)(4,7) = 2$, $|\phi(y^{-1})\phi(x)| = 2$, $|\phi(y^{-1})\phi(z)\phi(y)\phi(w)| = 1$,
   $|\phi(x)\phi(z)| = (2,6)(3,5) = 2$, $|\phi(z)\phi(w)| = (3,6)(4,7) = 2$, and
   $|\phi(y)\phi(z)\phi(y^{-1})\phi(w)| = 1$.
   Therefore, $\phi(N) \cong S_4$.

2. Now we compute $\phi(t)^{\phi(N)}$:
   $\phi(t)^e = (1,2)(3,4) = t_1$, $\phi(t)^{(\phi(z)\phi(y))} = (1,7)(3,5) = t_2$,
   $\phi(t)^{(\phi(x)\phi(w))} = (1,6)(4,5) = t_3$, $\phi(t)^{\phi(x)} = (1,3)(2,4) = t_4$,
   $\phi(t)^{(\phi(w)\phi(y))} = (1,7)(2,6) = t_5$, $\phi(t)^{\phi(z)} = (1,5)(4,6) = t_6$,
   $\phi(t)^{\phi(y)} = (1,4)(2,3) = t_7$, $\phi(t)^{\phi(w)\phi(x)} = (1,6)(2,7) = t_8$,
   $\phi(t)^{\phi(w)} = (1,5)(3,7) = t_9$, $\phi(t)^{\phi(x)\phi(w^{-1})} = (1,3)(5,7) = t_{10}$,
   $\phi(t)^{(\phi(z)\phi(x)\phi(y^{-1}))^3} = (1,4)(5,6) = t_{11}$, and $\phi(t)^{\phi(z)\phi(w)} = (1,2)(6,7) = t_{12}$.
   Therefore, $\phi(t)^{\phi(N)} = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}\}$.

3. Now we need to show that $\phi(N)$ acts as S_4 on the twelve conjugates of $\phi(t)$ by conjugates. Fist we conjugate by $\phi(x)$.
   $t_1^{\phi(x)} = (1,3)(2,4) = t_4$, $t_4^{\phi(x)} = (1,2)(3,4) = t_1$,
   $t_2^{\phi(x)} = (1,7)(2,6) = t_5$, $t_5^{\phi(x)} = (1,7)(3,5) = t_2$,
   $t_3^{\phi(x)} = (1,5)(4,6) = t_6$, $t_6^{\phi(x)} = (1,6)(4,5) = t_3$,
   $t_8^{\phi(x)} = (1,5)(3,7) = t_9$, $t_9^{\phi(x)} = (1,6)(2,7) = t_8$,
   $t_{10}^{\phi(x)} = (1,2)(6,7) = t_{12}$, $t_{12}^{\phi(x)} = (1,3)(5,7) = t_{10}$
   Thus, $\phi(x) = (t_1, t_4)(t_2, t_5)(t_3, t_6)(t_8, t_9)(t_{10}, t_{12})$.

We conjugate by $\phi(y)$.
$t_1^{\phi(y)} = (1,4)(2,3) = t_7$, $t_7^{\phi(y)} = (1,3)(2,4) = t_4$,
$t_4^{\phi(y)} = (1,2)(3,4) = t_1$,
$t_2^{\phi(y)} = (1,6)(2,7) = t_8$, $t_8^{\phi(y)} = (1,5)(4,6) = t_6$,
$t_6^{\phi(y)} = (1,7)(3,5) = t_2$,
$t_3^{\phi(y)} = (1,5)(3,7) = t_9$, $t_9^{\phi(y)} = (1,7)(2,6) = t_5$,
$t_5^{\phi(y)} = (1,6)(4,5) = t_3$,
$t_{10}^{\phi(y)} = (1,2)(6,7) = t_{12}$, $t_{12}^{\phi(y)} = (1,4)(5,6) = t_{11}$,
$t_{11}^{\phi(y)} = (1,3)(5,7) = t_{10}$.
Hence, $\phi(y) = (t_1, t_7, t_4)(t_2, t_8, t_6)(t_3, t_9, t_5)(t_{10}, t_{12}, t_{11})$.

We conjugate by $\phi(z)$.
$t_1^{\phi(z)} = (1,5)(4,6) = t_6$, $t_6^{\phi(z)} = (1,2)(3,4) = t_1$,
$t_2^{\phi(z)} = (1,7)(2,6) = t_5$, $t_5^{\phi(z)} = (1,7)(3,5) = t_2$,
$t_3^{\phi(z)} = (1,3)(2,4) = t_4$, $t_4^{\phi(z)} = (1,6)(4,5) = t_3$,
$t_7^{\phi(z)} = (1,4)(5,6) = t_{11}$, $t_{11}^{\phi(z)} = (1,4)(2,3) = t_7$,
$t_8^{\phi(z)} = (1,3)(5,7) = t_{10}$, $t_{10}^{\phi(z)} = (1,6)(2,7) = t_8$
$t_9^{\phi(z)} = (1,2)(6,7) = t_{12}$, $t_{12}^{\phi(z)} = (1,5)(3,7) = t_9$
Thus, $\phi(z) = (t_1, t_6)(t_2, t_5)(t_3, t_4)(t_7, t_{11})(t_8, t_{10})(t_9, t_{12})$.

Lastly, we conjugate by $\phi(w)$.
$t_1^{\phi(w)} = (1,5)(3,7) = t_9$, $t_9^{\phi(w)} = (1,2)(3,4) = t_1$,
$t_2^{\phi(w)} = (1,4)(2,3) = t_7$, $t_7^{\phi(w)} = (1,7)(3,5) = t_2$,
$t_3^{\phi(w)} = (1,6)(2,7) = t_8$, $t_8^{\phi(w)} = (1,6)(4,5) = t_3$,
$t_4^{\phi(w)} = (1,3)(5,7) = t_{10}$, $t_{10}^{\phi(w)} = (1,3)(2,4) = t_4$,
$t_5^{\phi(w)} = (1,4)(5,6) = t_{11}$, $t_{11}^{\phi(w)} = (1,7)(2,6) = t_5$
$t_6^{\phi(w)} = (1,2)(6,7) = t_{12}$, $t_{12}^{\phi(w)} = (1,5)(4,6) = t_6$
Thus, $\phi(w) = (t_1, t_9)(t_2, t_7)(t_3, t_8)(t_4, t_{10})(t_5, t_{11})(t_6, t_{12})$.

Therefore, $\phi(G) = \langle \phi(x), \phi(y), \phi(z), \phi(w), \phi(t) \rangle$ is a homomorphic image of $2^{*12} : S_4$.

By the First Isomorphism Theorem we have:

$$G/Ker\phi \cong \phi(G)$$

$$\implies |G/Ker\phi| \cong |\phi(G)|$$

$$\implies |G| = |Ker\phi||\phi(G)|.$$

If we look back to the double coset enumeration we know $|G| \leq 168$. Furthermore, by MAGMA we were able to calculate $|\phi(G)|$.

$$|\phi(G)| = |\langle \phi(x), \phi(y), \phi(z), \phi(w), \phi(t) \rangle| = 168.$$

$$|G| = |Ker\phi|168$$

$$\implies |G| \geq 168.$$

So, $|Ker\phi| = 1$.
Therefore, $|G| = 168$.

4.2.3 Finding Generators of $PSL(2, 7)$

Definition 4.1. *[Cur07] To find generators of $PSL(n, q)$ and $PGL(n, q)$, where $q = p^n$. Every finite field is of order p^n, where p is prime.*

$$\textbf{Example}: \ \mathbf{PSL(2,7)} \cong$$

$$\langle x, y, z, w, t | x^2, y^3, z^2, w^2, (y^{-1}x)^2, y^{-1}zyw, (xz)^2, (zw)^2, yz * y^{-1}zw,$$

$$t^2,$$

$$(t, xy), (xy^{-1}wt)^3, (yt)^3, (wt)^3, xwytzwt, tztxwytwt \rangle$$

where the control group $N \cong S_4 =$

$$\langle x, y, z, w, | x^2, y^3, z^2, w^2, (y^{-1}x)^2, y^{-1}zyw, (xz)^2, (zw)^2, yz * y^{-1}zw \rangle.$$

So we have the following:

$$G \cong \frac{2^{*12} : S_4}{(yt)^3, (wt)^3, zxwywt_{12}t_6 = t_1t_9, xwyzwt_{12} = t_1} \cong PSL(2,7).$$

Let $x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$, $y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$,
$z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$, and $w \sim (1,9)(2,7)(3,8)(4,10)(5,11)(6,12)$
where $t \sim t_1$.
We construct a homomorphism ϕ from the progenitor $2^{*12} : S_4$ to $PSL_2(7)$ by defining $\phi(x)$, $\phi(y)$, $\phi(z)$, $\phi(w)$, and $\phi(t_1)$.

Consider the field of order seven where $F_7 = \{0,1,2,3,4,5,6\}$ and $F_7 \cup (\infty)$ over twelve letters.
Now we find the Nonzero Squares of F_7. We get:

$$\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2\}$$

$$= \{1,4,2,2,4,1\}$$

$$= \{1,2,4\}.$$

<u>Facts about finite field</u> :

$$G = F_7 - \{0\} \text{ is cyclic}$$

$$= \{a^2 | a \in G\} \leq G$$

$$\Rightarrow \{a^2 | a \in G\} \text{ is cyclic and k is a generator of the group.}$$

$PSL_2(7) = \{x \longmapsto \frac{ax+b}{cx+d}$, where $a,b,c,d \in F_7, x \in F_7 \cup \{\infty\} | ad - bc = 1$
or a nonzero square$\}$

$$= \langle \alpha, \beta, \gamma \rangle.$$

Now we find α, β, and γ as follows:
$\alpha : x \longmapsto x + 1$ where $x \in F_7 \cup \{\infty\}$

$$= (0,1,2,3,4,5,6), (\infty).$$

$\beta : x \longmapsto kx$ where k is a nonzero square whose powers gives all of the nonzero squares. Now check for a nonzero square to get k as follows

$$4^1 = 4$$

$$4^2 = 16 = 2(mod7)$$

$$4^3 = 64 = 1(mod7).$$

Thus $k = 4$.
Since $k = 4$ we have $\beta : x \longmapsto 4x$ where $x \in F_7 \cup \{\infty\}$.
So we get $\beta = (0)(\infty)(1, 4, 2)(3, 5, 6)$.

$\gamma : x \longmapsto \frac{ax+b}{cx+d}$ where $a, b, c, d \in F_7$ and $ad - bc = 1$ or equivalent to a nonzero square.
Since $x \longmapsto -\frac{1}{x}$ where $x \in F_7 \cup \{\infty\}$, then
$\gamma = (0, \infty)(1, 6)(2, 3)(4, 5)$.
We now show that G is isomorphic to $L_2(7)$ and construct a homomorphism from the progenitor $2^{*12} : S_4$ to $L_2(7)$ by defining $\phi(x)$, $\phi(y)$, $\phi(z)$, $\phi(w)$ and $\phi(t_1)$.
We use the the following group where $G \cong PSL_2(7)$ with generators, x,y z, and w . We now store alpha, beta, and gamma. Note, $PSL_2(7)$ is symmetric in eight letters where $7 \sim 0$ and $8 \sim \infty$.

```
S:=Sym(8);
alpha:=S!(7,1,2,3,4,5,6);
beta:=S!(1,4,2)(3,5,6);
gamma:=S!(7,8)(1,6)(2,3)(4,5);
psl:=sub<S|alpha, beta, gamma>;
IsIsomorphic(G1,psl);
/*true Homomorphism of GrpPerm: G1, Degree 7,
Order 2^3 * 3 * 7 into GrpPerm: psl,
Degree 8, Order 2^3 * 3 * 7 induced by
    (2, 3)(5, 6) |--> (1, 7)(2, 4)(3, 8)(5, 6)
    (2, 4, 3)(5, 7, 6) |--> (1, 6, 2)(4, 5, 7)
    (2, 5)(3, 6) |--> (1, 8)(2, 6)(3, 7)(4, 5)
    (2, 5)(4, 7) |--> (1, 2)(3, 4)(5, 7)(6, 8)
    (1, 2)(3, 4) |--> (1, 3)(2, 6)(4, 8)(5, 7)
```

After we have stored alpha, beta, and gamma we ask MAGMA if $G1$ is isomorphic to

$PSL(2,7)$, true. On the left hand side of the mapping we have $\phi(x)$,$\phi(y)$, $\phi(z)$, $\phi(w)$, and $\phi(t_1)$ of G. On the right hand side of the mapping we have the homomorphisms of $L_2(7)$.

To find a map that satisfies each homomorphism of $L_2(7)$ we do so as follows:

For example, the homomorphism of $\phi(x) = (1,0)(2,4)(3,\infty)(5,6)$ is obtained as follows:

$$(*)\ \underline{0 \longmapsto 1} \text{ using } x \longmapsto \frac{ax+b}{cx+d}.$$

$$\frac{a(0)+b}{c(o)+d} = 1 \Rightarrow \frac{b}{d} = 1 \Rightarrow b = d,$$

$$(*)\ \underline{1 \longmapsto 0}$$

$$\frac{a(1)+b}{c(1)+d} = 0 \Rightarrow a+b = 0 \Rightarrow a = -b,$$

$$(*)\ \underline{2 \longmapsto 4}$$

$$\frac{a(2)+b}{c(2)+d} = 4 \Rightarrow 2a+b = c+4d \Rightarrow -2b+b-4b = c \Rightarrow c = 2b$$

We now replace a, c, and d to $x \longmapsto \frac{ax+b}{cx+d} =$
$\frac{-bx+b}{2bx+b} = \frac{\cancel{b}^1(-x+1)}{\cancel{b}^1(2x+1)} = \frac{-x+1}{2x+1}$.

Thus, we have defined the homomorphism of $\phi(x)$, we use the same method to defined $\phi(y)$, $\phi(z)$, $\phi(w)$ and $\phi(t_1)$, respectively.

$$\phi(x) \equiv \left(\frac{-x+1}{2x+1}\right) = (1,0)(2,4)(3,\infty)(5,6),$$

$$\phi(y) \equiv 2x+4 = (1,6,2)(4,5,0),$$

$$\phi(z) \equiv \left(\frac{x+4}{x+6}\right) = (1,\infty)(2,6)(3,0)(4,5), \text{and}$$

$$\phi(w) \equiv \left(\frac{x+2}{6x+6}\right) = (1,2)(3,4)(5,0)(6,\infty).$$

Since the orders of $\phi(x)$, $\phi(y)$, $\phi(z)$, $\phi(w)$, $\phi(y^{-1}x)$, $\phi(xz)$, $\phi(zw)$ are 2, 3, 2, 2, 2, 2, 2, respectively, $\mathcal{N} \cong S_4$.
We continue by defining $\phi(t_1)$, which is

$$\phi(t_1) \equiv \left(\frac{x+2}{2x+6}\right) = (1,3)(2,6)(4,\infty)(5,0).$$

We have verified that $PSL_2(7) \cong \langle\phi(x), \phi(y), \phi(z), \phi(w), \phi(t_1)\rangle$. Now we show that ϕ preserves the operation of $2^{*12} : S4$.
We find the order of $\phi(t_1)^{\phi(\mathcal{N})}$, $|\phi(t_1)^{\phi(\mathcal{N})}| = 12$.
We now compute $\phi(t_1)^{\mathcal{N}}$ to get all twelve $t's$ and

$$\phi(t) = \phi(t_1) = (1,3)(2,6)(4,8)(5,7),$$

$$\phi(t_1^x) = \phi(t_4) = (1,6)(2,3)(4,5)(7,8),$$

$$\phi(t_1^{z*y}) = \phi(t_2) = (1,2)(3,5)(4,8)(6,7),$$

$$\phi(t_2^x) = \phi(t_5) = (1,5)(2,3)(4,7)(6,8),$$

$$\phi(t_1^{x^w}) = \phi(t_3) = (1,3)(2,8)(4,5)(6,7),$$

$$\phi(t_3^x) = \phi(t_6) = (1,5)(2,6)(3,4)(7,8),$$

$$\phi(t_1^y) = \phi(t_7) = (1,2)(3,6)(4,7)(5,8),$$

$$\phi(t_1^{xw}) = \phi(t_8) = (1,6)(2,4)(3,7)(5,8),$$

$$\phi(t_8^x) = \phi(t_9) = (1,8)(2,4)(3,6)(5,7),$$

$$\phi(t_1^{xw^{-1}}) = \phi(t_{10}) = (1,4)(2,8)(3,7)(5,6),$$

$$\phi(t_1^{(zxy^{-1})^3}) = \phi(t_{11}) = (1,4)(2,7)(3,5)(6,8),$$

$$\phi(t_{10}^x) = \phi(t_{12}) = (1,8)(2,7)(3,4)(5,6).$$

Thus, $\phi(t^N) = \{\phi(t_1), \phi(t_2), \phi(t_3), \phi(t_4), \phi(t_5), \phi(t_6), \phi(t_7), \phi(t_8), \phi(t_9), \phi(t_{10}), \phi(t_{11}), \phi(t_{12})\}$. We continue by checking that $\phi(\mathcal{N})$ permutes the twelve images of $\phi(t_1)$, by conjugation, as the group S_4 given by:

First we check $\phi(t)^{\phi(x)}$:

$\phi(t)^{\phi(x)} = (1,6)(2,3)(4,5)(7,8) = \phi(t_4)$, $\phi(t_4)^{\phi(x)} = (1,3)(2,6)(4,8)(5,0) = \phi(t_1)$,
$\phi(t_2)^{\phi(x)} = (1,5)(2,3)(4,7)(6,8) = \phi(t_5)$, $\phi(t_5)^{\phi(x)} = (1,2)(3,5)(4,8)(6,7) = \phi(t_2)$,
$\phi(t_3)^{\phi(x)} = (1,5)(2,6)(3,4)(7,8) = \phi(t_6)$, $\phi(t_6)^{\phi(x)} = (1,3)(2,8)(4,5)(6,7) = \phi(t_3)$,
$\phi(t_7)^{\phi(x)} = (1,2)(3,6)(4,7)(5,8) = \phi(t_7)$, $\phi(t_{11})^{\phi(x)} = (1,4)(2,7)(3,5)(6,8) = \phi(t_{11})$,
$\phi(t_8)^{\phi(x)} = (1,8)(2,4)(3,6)(5,7) = \phi(t_9)$, $\phi(t_9)^{\phi(x)} = (1,6)(2,4)(3,7)(5,8) = \phi(t_8)$,
$\phi(t_{10})^{\phi(x)} = (1,8)(2,7)(3,4)(5,6) = \phi(t_{12})$, $\phi(t_{12})^{\phi(x)} = (1,4)(2,8)(3,7)(5,6) = \phi(t_{10})$

Thus, $\phi(x) = (\phi(t_1), \phi(t_4))(\phi(t_2), \phi(t_5))(\phi(t_3), \phi(t_6))(\phi(t_8), \phi(t_9))(\phi(t_{10}), \phi(t_{12}))$.

We conjugate by $\phi(y)$.

$\phi(t_1)^{\phi(y)} = (1,2)(3,6)(4,7)(5,8) = \phi(t_7)$, $\phi(t_7)^{\phi(y)} = (1,6)(2,3)(4,5)(7,8) = \phi(t_4)$,
$\phi(t_4)^{\phi(y)} = (1,3)(2,6)(4,8)(5,7) = \phi(t_1)$,
$\phi(t_2)^{\phi(y)} = (1,6)(2,4)(3,7)(5,8) = \phi(t_8)$, $\phi(t_8)^{\phi(y)} = (1,5)(2,6)(3,4)(7,8) = \phi(t_6)$,
$\phi(t_6)^{\phi(y)} = (1,2)(3,5)(4,8)(6,7) = \phi(t_2)$,
$\phi(t_3)^{\phi(y)} = (1,8)(2,4)(3,6)(5,7) = \phi(t_9)$, $\phi(t_9)^{\phi(y)} = (1,5)(2,3)(4,7)(6,8) = \phi(t_5)$,
$\phi(t_5)^{\phi(y)} = (1,3)(2,8)(4,5)(6,7) = \phi(t_3)$,
$\phi(t_{10})^{\phi(y)} = (1,8)(2,7)(3,4)(5,6) = \phi(t_{12})$, $\phi(t_{12})^{\phi(y)} = (1,4)(2,7)(3,5)(6,8) = \phi(t_{11})$,
$\phi(t_{11})^{\phi(y)} = (1,4)(2,8)(3,7)(5,6) = \phi(t_{10})$.

Hence, $\phi(y) = (\phi(t_1), \phi(t_7), \phi(t_4))\,(\phi(t_2), \phi(t_8), \phi(t_6))(\phi(t_3), \phi(t_9), \phi(t_5))$
$(\phi(t_{10}), \phi(t_{12}), \phi(t_{11}))$.

We conjugate by $\phi(z)$.

$\phi(t_1)^{\phi(z)} = (1,5)(2,6)(3,4)(7,8) = \phi(t_6)$, $t_6^{\phi(z)} = (1,3)(2,6)(4,8)(5,7) = \phi(t_1)$,
$\phi(t_2)^{\phi(z)} = (1,5)(2,3)(4,7)(6,8) = \phi(t_5)$, $t_5^{\phi(z)} = (1,2)(3,5)(4,8)(6,7) = \phi(t_2)$,
$\phi(t_3)^{\phi(z)} = (1,6)(2,3)(4,5)(7,8) = \phi(t_4)$, $t_4^{\phi(z)} = (1,3)(2,8)(4,5)(6,7) = \phi(t_3)$,
$\phi(t_7)^{\phi(z)} = (1,4)(2,7)(3,5)(6,8) = \phi(t_{11})$, $t_{11}^{\phi(z)} = (1,2)(3,6)(4,7)(5,8) = \phi(t_7)$,
$\phi(t_8)^{\phi(z)} = (1,4)(2,8)(3,7)(5,6) = \phi(t_{10})$, $t_{10}^{\phi(z)} = (1,6)(2,4)(3,7)(5,8) = \phi(t_8)$
$\phi(t_9)^{\phi(z)} = (1,8)(2,7)(3,4)(5,6) = \phi(t_{12})$, $t_{12}^{\phi(z)} = (1,8)(2,4)(3,6)(5,7) = \phi(t_9)$

Thus, $\phi(z) = (\phi(t_1), \phi(t_6))(\phi(t_2), \phi(t_5))(\phi(t_3), \phi(t_4))(\phi(t_7), \phi(t_{11}))(\phi(t_8), \phi(t_{10}))$

$(\phi(t_9), \phi(t_{12}))$.

Lastly, we conjugate by $\phi(w)$.
$\phi(t_1)^{\phi(w)} = (1,8)(2,4)(3,6)(5,7) = \phi(t_9)$, $\phi(t_9)^{\phi(w)} = (1,3)(2,6)(4,8)(5,7) = \phi(t_1)$,
$\phi(t_2)^{\phi(w)} = (1,2)(3,6)(4,7)(5,8) = \phi(t_7)$, $\phi(t_7)^{\phi(w)} = (1,2)(3,5)(4,8)(6,7) = \phi(t_2)$,
$\phi(t_3)^{\phi(w)} = (1,6)(2,4)(3,7)(5,8) = \phi(t_8)$, $\phi(t_8)^{\phi(w)} = (1,3)(2,8)(4,5)(6,7) = \phi(t_3)$,
$\phi(t_4)^{\phi(w)} = (1,4)(2,8)(3,7)(5,6) = \phi(t_{10})$, $\phi(t_{10})^{\phi(w)} = (1,6)(2,3)(4,5)(7,8) = \phi(t_4)$,
$\phi(t_5)^{\phi(w)} = (1,4)(2,7)(3,5)(6,8) = \phi(t_{11})$, $\phi(t_{11})^{\phi(w)} = (1,5)(2,3)(4,7)(6,8) = \phi(t_5)$
$\phi(t_6)^{\phi(w)} = (1,8)(2,7)(3,4)(5,6) = \phi(t_{12})$, $\phi(t_{12})^{\phi(w)} = (1,5)(2,6)(3,4)(7,8) = \phi(t_6)$
Thus, $\phi(w) = (\phi(t_1), \phi(t_9))(\phi(t_2), \phi(t_7))(\phi(t_3), \phi(t_8))(\phi(t_4), \phi(t_{10}))(\phi(t_5), \phi(t_{11}))$
$(\phi(t_6), \phi(t_{12}))$.
Therefore, $\phi(2^{*12} : S_4) = PSL_2(7)$.

Now the additional relations given by $(yt)^3 = 1 \iff t_1t_7t_4 = Id(G)$ is satisfied in $L_2(7)$, since $t_1t_7t_4 = Id(G)$ acts as $Id(G)$ by conjugation on the twelve symmetric generators, $(wt)^3 = 1 \iff t_1t_9t_1 = w^3 = (1,9)(2,7)(3,8)(4,10)(5,11)(6,12)$ is satisfies in $L_2(7)$, since $t_1t_9t_1 = (1,2)(3,4)(5,7)(6,8)$ acts as $(t_1, t_9)(t_2, t_7)(t_3, t_8)(t_4, t_{10})(t_5, t_{11})$ (t_6, t_{12}) by conjugation on the twelve symmetric generators, $zxwywt_{12}t_6t_9t_1 = 1 \iff$ $t_1t_9t_6t_{12} = zxwyw = (1,12)(2,10)(3,11)(4,5)(7,8)$ is satisfied in $L_2(7)$, since $t_1t_9t_6t_{12} =$ $(1,7)(2,3)(4,6)(5,8)$ acts as $(t_1, t_{12})(t_2, t_{10})(t_3, t_{11})(t_4, t_5)(t_7, t_8)$ by conjugations on the twelve symmetric generators, and $xwyzwt_{12}t_1 = 1 \iff t_1t_{12} = xwyzw = (2,3)(4,7)$ $(5,8)(6,9)(10,11)$ is satisfied in $L_2(7)$, since $t_1t_{12} = (1,4)(2,5)(3,8)(6,7)$ acts as (t_2, t_3) $(t_4, t_7)(t_5, t_8)(t_6, t_9)(t_{10}, t_{11})$ by conjugation on the twelve symmetric generators. We have shown that $L_2(7)$ is an image of G.
Hence, $|G| \geq |L_2(7)|$ but by the cayley graph we know $|G| \leq 168 = |L_2(7)|$ and so the equality holds.
Therefore, $G \cong L_2(7)$.

4.3 Construction of $PGL_2(13)$ over S_4

4.3.1 Double Coset Enumeration of G

Consider the group $G = 2^{*12} : S_4$ factored by the relators $(xy^{-1}z^{y^{-2}}t)^7$ and $(y * t)^2$ with symmetric presentation given by

$$\langle x, y, z, t | x^2, y^3, z^2, (y^{-1}x)^2, y^{-1}zy(z^{(y^{-2})}), (xz)^2, (z(z^{(y^{-2})}))^2, yzy^{-1}z(z^{(y^{-2})}),$$

$$t^2,$$

$$(t, xy), (xy^{-1}(z^{(y^{-2})})t)^7, (yt)^2\rangle$$

where $N \cong S_4 =$

$$\langle x^2, y^3, z^2, (y^{-1}x)^2, y^{-1}zy(z^{(y^{-2})}), (xz)^2, (z(z^{(y^{-2})}))^2, yzy^{-1}z(z^{(y^{-2})})\rangle.$$

Then,

$$G \cong \frac{2^{*12} : S_4}{(xy^{-1}z^{y^{-2}}t)^7, (yt)^2} \cong PGL(2, 13)).$$

Note $N = S_4$, where $x \sim (1, 4)(2, 5)(3, 6)(8, 9)(10, 12)$,
$y \sim (1, 7, 4)(2, 8, 6)(3, 9, 5)(10, 12, 11)$ and $z \sim (1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12)$. Let $t \sim t_1$. Let us expand the relations:

Let $\pi = xy^{-1}z^{y^{-2}} = (1, 2)(4, 10)(5, 12)(6, 11)(7, 9)$ and

$$(\pi t)^7 = (\pi t_1)^7 = 1$$

$$(\pi t_1)^7 = 1$$

$$\implies (\pi t_1)^7 = \pi^7 t_1^{\pi^6} t_1^{\pi^5} t_1^{\pi^4} t_1^{\pi^3} t_1^{\pi^2} t_1^{\pi} t_1 = 1$$

$$\implies \pi^7 t_1^{\pi^6} t_1^{\pi^5} t_1^{\pi^4} t_1^{\pi^3} t_1^{\pi^2} t_1^{\pi} t_1 = 1$$

$$(1, 2)(4, 10)(5, 12)(6, 11)(7, 9)t_1t_2t_1t_2t_1t_2t_1 = 1$$

$$\implies t_1t_2 = (1, 2)(4, 10)(5, 12)(6, 11)(7, 9)t_1t_2t_1t_2t_1$$

and

$y = (1, 7, 4)(2, 8, 6)(3, 9, 5)(10, 12, 11)$

$$\implies (yt)^2 = y^2 t_1^y t_1 = 1$$

$$y^2 t_1^y t_1 = 1$$

$$\implies (1, 4, 7)(2, 6, 8)(3, 5, 9)(10, 11, 12)t_7 t_1 = 1$$

$$\implies t_1 = (1, 4, 7)(2, 6, 8)(3, 5, 9)(10, 11, 12)t_7$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N.

NeN

First double coset NeN, is denoted by [$*$]. This double coset contains only the single coset, namely N. Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, the orbit of N on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is: $\mathbb{O} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
Let t_1 be our symmetric generator from this orbit $\mathbb{O}$ and find to which double coset Nt_1 belongs. Nt_1 will be a new double coset, denoted by [1], so twelve symmetric generators will go to [1].

Nt*₁*N

In order to find how many single cosets [1] contains, we must first find the coset stabiliser, $N^{(1)}$. Now, $N^1 = \langle e, (2, 3)(4, 7)(5, 8)(6, 9)(10, 11)\rangle$, but the coset stabaliser of $1 \in N$ is $N^{(1)} \geq \langle e, (2, 3)(4, 7)(5, 8)(6, 9)(10, 11), (1, 4, 7)(2, 6, 8)(3, 5, 9)(10, 11, 12),$ $(1, 4)(2, 5)(3, 6)(8, 9)(10, 12)\rangle$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. So the number of the single cosets in Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{24}{6} = 4$ since there are equal names by $t_1 = yt_4$. The orbits of N^1 on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are:

$$\mathbb{O} = \{1\}, \{12\}, \{2, 3\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \text{and}\{10, 11\}$$

We take t_1, t_{12}, t_2, t_4,t_5 and t_6 from each orbit, respectively, and find to which double coset
Nt_1t_1, Nt_1t_{12}, Nt_1t_2, Nt_1t_4, Nt_1t_5 and Nt_1t_6 belong to. Now $Nt_1t_1 = N \in [*]$, so one element will go back to [$*$]. On the other hand, one symmetric generator will go to a new double coset Nt_1t_{12} denoted by [112]. But the following single cosets live in

other double cosets,$Nt_1t_2 \in [112]$ by $t_1t_2 = y^{-1}zt_6t_9$, $Nt_1t_5 \in [112]$ by $t_1t_5 = y^{-1}zt_3t_8$, $Nt_1t_6 \in [112]$ by $t_1t_6 = yzy^{-1}zt_8t_3$, $Nt_1t_{10} \in [112]$ by $t_1t_{10} = yt_4t_{10}$ so for each single coset two representatives go to [112], and $Nt_1t_4 = Nt_1t_1 \in [*]$ (since $t_1 = t_4$) two representatives go to [∗].

$\boldsymbol{Nt_1t_{12}N}$

Now $Nt_1t_{12}N$ in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(112)} \geq \langle Id(N), (2,3)(4,7)(5,8)(6,9)(10,11)\rangle$. Then, $|N^{(112)}| = 2$ so the number of single cosets in $N^{(112)}$ is $\frac{|N|}{|N^{(112)}|} = \frac{24}{2} = 12$. Now $Nt_1t_{12}N$ is indeed a new double coset. We determine the twelve single coset names in [112]: are $\{Nt_1t_{12}, Nt_4t_{10}, Nt_7t_{11}, Nt_6t_9, Nt_3t_8, N_{11}t_7, Nt_2t_5, Nt_9t_6, Nt_{10}t_4, Nt_5t_2, Nt_8t_3, Nt_{12}t_1\}$
The orbits of $N^{(112)}$ on {1,2,3,4,5,6,7,8,9,10,11,12} are:
$\mathbb{O} = \{1\}, \{12\}, \{2,3\}, \{4,7\}, \{5,8\}, \{6,9\}, \text{and}\{10,11\}.$
Take a representative t_i from each orbit and see which double coset $Nt_1t_{12}t_i$ belongs to. We have:

$$
\begin{aligned}
Nt_1t_{12}t_1 &=\in [1121]\\
Nt_1t_{12}t_{12} &= Nt_1 \in [1]\\
Nt_1t_{12}t_2 &\in [1122]\\
Nt_1t_{12}t_4 &= yt_7t_{11}t_7\\
\implies Nt_1t_{12}t_4 &= Nt_7t_{11}t_7 \in [1121]\\
Nt_1t_{12}t_5 &= y^{-1}zt_3\\
\implies Nt_1t_{12}t_5 &= Nt_3 \in [1]\\
Nt_1t_{12}t_6 &= yzyt_{10}t_4t_{11}\\
\implies Nt_1t_{12}t_6 &= Nt_{10}t_4t_{11} \in [1122]\\
t_1t_{12}t_{10} &= zy^{-1}t_8t_3t_9\\
\implies Nt_1t_{12}t_{10} &= Nt_8t_3t_9 \in [1122]
\end{aligned}
$$

The new double coset $Nt_1t_{12}t_1N$ is denoted by [1 12 1] and $Nt_1t_{12}t_1$ is its representative right coset. Also $Nt_1t_{12}t_2N$ is denoted by [1122] and $Nt_1t_{12}t_2$ is its representative right coset.

$Nt_1t_{12}t_1N$

Now $Nt_1t_{12}t_1N$ in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(1121)} \geq \langle Id(N), (2,3)(4,7)(5,8)(6,9)(10,11)\rangle$. Then, $|N^{(1121)}| = 2$ so the number of single cosets in $N^{(1121)}$ is $\frac{|N|}{|N^{(1121)}|} = \frac{24}{2} = 12$. The twelve single cosets in [1121] are $\{Nt_1t_{12}t_1, Nt_4t_{10}t_4, Nt_7t_{11}t_7, Nt_6t_9t_6, Nt_3t_8t_3, Nt_2t_5t_2,$ $Nt_5t_2t_5, Nt_8t_3t_8, Nt_9t_6t_9, Nt_{11}t_7t_{11}, Nt_{10}t_4t10, Nt_{12}t_1t_{12}\}$. The orbits of $N^{(1121)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are: $\mathbb{O} = \{1\}, \{12\}, \{2,3\}, \{4,7\}, \{5,8\}, \{6,9\}$, and$\{10,11\}$. Take a representative t_i from each orbit and right multiply to the representative right coset $Nt_1t_{12}t_1t_i$. We have

$$Nt_1t_{12}t_1t_1 = Nt_1t_{12} \in [112]$$

$$t_1t_{12}t_1t_{12} = xzyzt_1t_{12}t_1$$

$$\implies Nt_1t_{12}t_1t_{12} = Nt_1t_{12}t_1 \in [1121]$$

$$Nt_1t_{12}t_1t_2 \in [11212]$$

$$t_1t_{12}t_1t_4 = yt_7t_{11}$$

$$\implies Nt_1t_{12}t_1t_4 = Nt_7t_{11} \in [112]$$

$$t_1t_{12}t_1t_5 = xyzy^{-1}t_3t_8t_3$$

$$\implies Nt_1t_{12}t_1t_5 = Nt_3t_8t_3 \in [1121]$$

$$t_1t_{12}t_1t_6 = yzyt_{10}t_4t_{10}t_{11}$$

$$\implies Nt_1t_{12}t_1t_6 = Nt_{10}t_4t_{10}t_{11} \in [11212]$$

$$t_1t_{12}t_1t_{10} = zy^{-1}t_8t_3t_8t_9$$

$$\implies Nt_1t_{12}t_1t_{10} = Nt_8t_3t_8t_9 \in [11212]$$

The new double coset $Nt_1t_{12}t_1t_2N$ is denoted by [1 12 1 2] and $Nt_1t_{12}t_1t_2$ is its representative right coset.

$Nt_1t_{12}t_2N$

Now $Nt_1t_{12}t_2N$ is a new double coset. However, $N^{(1122)} = N^{1122} = \langle Id(N)\rangle$. Only identity e will fix 1, 12 and 2. Therefore the number of the single cosets in $Nt_1t_{12}t_2N$ is $\frac{|N|}{|N^{(1122)}|} = \frac{24}{1} = 24$ and the names are $\{Nt_1t_{12}t_2, Nt_4t_{10}t_5, Nt_7t_{11}t_8, Nt_6t_9t_5, Nt_1t_{12}t_3,$ $Nt_3t_8t_2, Nt_7t_{11}t_9, Nt_4t_{10}t_6, Nt_{11}t_7t_{10}, Nt_2t_5t_3, Nt_6t_9t_4, Nt_9t_6t_8, Nt_{11}t_7t_{12}, Nt_3t_8t_1,$

$Nt_{10}t_4t12, Nt_5t_2t_6, Nt_8t_3t_9, Nt_5t_2t_4, Nt_2t_5t_1, Nt_{12}t_1t_{10}, Nt_{10}t_4t11, Nt_9t_6t_7, Nt_{12}t_1t_7,$ $Nt_8t_3t_7\}$. The orbits of $N^{(1122)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_{12}t_2t_i$ belongs to.We have the following:

$$
\begin{aligned}
t_1t_{12}t_2t_1 &\in [11221] \\
Nt_1t_{12}t_2t_2 &= Nt_1t_{12} \in [112] \\
Nt_1t_{12}t_2t_3 &= xzyzt_{10}t_4t_{12} \\
\implies Nt_1t_{12}t_2t_3 &= Nt_{10}t_4t_{12} \in [1122] \\
t_1t_{12}t_2t_4 &= y^{-1}zt_9t_6t_7t_9 \\
\implies Nt_1t_{12}t_2t_4 &= Nt_9t_6t_7t_9 \in [11221] \\
t_1t_{12}t_2t_5 &= yzt_5t_2t_5t_6 \\
\implies Nt_1t_{12}t_2t_5 &= Nt_5t_2t_5t_6 \in [11212] \\
t_1t_{12}t_2t_6 &= xt_5t_2t_4 \\
\implies Nt_1t_{12}t_2t_6 &= Nt_5t_2t_4 \in [1122] \\
t_1t_{12}t_2t_7 &= z^yt_5t_2t_4t_5 \\
\implies Nt_1t_{12}t_2t_7 &= Nt_5t_2t_4t_5 \in [11221] \\
t_1t_{12}t_2t_8 &= z^yt_{12}t_1t_{12}t_{10} \\
\implies Nt_1t_{12}t_2t_8 &= Nt_{12}t_1t_{12}t_{10} \in [11212] \\
Nt_1t_{12}t_2t_9 &= yzt11t_7 \\
\implies Nt_1t_{12}t_2t_9 &= Nt_{11}t_7 \in [112] \\
t_1t_{12}t_2t_{10} &= zy^{-1}t_8t_3 \\
\implies Nt_1t_{12}t_2t_{10} &= Nt_8t_3 \in [112] \\
t_1t_{12}t_2t_{11} &= xyzyt_1t_{12}t_2 \\
\implies Nt_1t_{12}t_2t_{11} &= Nt_1t_{12}t_2 \in [1122] \\
t_1t_{12}t_2t_{12} &= y^{-1}t_8t_3t_8t_7 \\
\implies Nt_1t_{12}t_2t_{12} &= Nt_8t_3t_8t_7 \in [11212]
\end{aligned}
$$

The new double coset $Nt_1t_{12}t_2t_1N$ is denoted by [1 12 2 1] and $Nt_1t_{12}t_2t_1$ is its representative right coset.

$Nt_1t_{12}t_1t_2N$

Now $Nt_1t_{12}t_1t_2N$ is a new double coset. However, $N^{(11212)} = N^{11212} = \langle Id(N)\rangle$. Only identity e will fix 1, 12, 1, and 2. Therefore the number of the single cosets in $Nt_1t_{12}t_1t_2N$ is $\frac{|N|}{|N^{(11212)}|} = \frac{24}{1} = 24$ and the names are $\{Nt_1t_{12}t_1t_2, Nt_4t_{10}t_4t_5,$ $Nt_7t_{11}t_7t_8, Nt_6t_9t_6t_5, Nt_1t_{12}t_1t_3, Nt_3t_8t_3t_2, Nt_7t_{11}t_7t_9,$ $Nt_4t_{10}t_4t_6, Nt_{11}t_7t_{11}t_{10}, Nt_2t_5t_2t_3, Nt_6t_9t_6t_4, Nt_9t_6t_9t_8,$ $Nt_{11}t_7t11t_{12}, Nt_3t_8t_3t_1, Nt_{10}t_4t_{10}t_{12}, Nt_5t_2t_5t_6, Nt_8t_3t_8t_9,$ $Nt_5t_2t_5t_4, Nt_2t_5t_2t_1, Nt_{12}t_1t_{12}t_{10}, Nt_{10}t_4t_{10}t_{11}, Nt_9t_6t_9t_7,$ $Nt_{12}t_1t_{12}t_{11}, Nt_8t_3t_8t_7\}$. The orbits of $N^{(11212)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_{12}t_1t_2t_i$ belongs to.

We have the following:

$$
\begin{aligned}
t_1t_{12}t_1t_2t_1 &= zyt_{11}t_7t_{12} \\
\implies Nt_1t_{12}t_1t_2t_1 &= Nt_{11}t_7t_{12} \in [1122] \\
Nt_1t_{12}t_1t_2t_2 &= Nt_1t_{12}t_1 \in [1121] \\
Nt_1t_{12}t_1t_2t_3 &\in [112123] \\
t_1t_{12}t_1t_2t_4 &= zy^{-1}t_{10}t_4t_{11} \\
\implies Nt_1t_{12}t_1t_2t_4 &= Nt_{10}t_4t_{11} \in [1122] \\
t_1t_{12}t_1t_2t_5 &= xyzt_3t_8t_3t_1 \\
\implies Nt_1t_{12}t_1t_2t_5 &= Nt_3t_8t_3t_1 \in [11212] \\
t_1t_{12}t_1t_2t_6 &= zyzt_1t12t_1t_3t_2 \\
\implies Nt_1t_{12}t_1t_2t_6 &= Nt_1t12t_1t_3t_2 \in [112123] \\
t_1t_{12}t_1t_2t_7 &= zt_{12}t_1t_{10} \\
\implies Nt_1t_{12}t_1t_2t_7 &= Nt_{12}t_1t_{10} \in [1122] \\
t_1t_{12}t_1t_2t_8 &= xy^{-1}zyt_2t_5t_2t_3 \\
\implies Nt_1t_{12}t_1t_2t_8 &= Nt_2t_5t_2t_3 \in [11212]
\end{aligned}
$$

$$t_1t_{12}t_1t_2t_9 = yzt_{11}t_7t_{11}$$
$$\implies Nt_1t_{12}t_1t_2t_9 = Nt_{11}t_7t_{11} \in [1121]$$
$$t_1t_{12}t_1t_2t_{10} = zy^{-1}t_8t_3t_8$$
$$\implies Nt_1t_{12}t_1t_2t_{10} = Nt_8t_3t_8 \in [1121]$$
$$t_1t_{12}t_1t_2t_{11} = zyzt_5t_2t_5t_4t_6$$
$$\implies Nt_1t_{12}t_1t_2t_{11} = Nt_5t_2t_5t_4t_6 \in [112123]$$
$$t_1t_{12}t_1t_2t_{12} = xy^{-1}t_1t_{12}t_1t_2$$
$$\implies Nt_1t_{12}t_1t_2t_{12} = Nt_1t_{12}t_1t_2 \in [11212]$$

The new double coset $Nt_1t_{12}t_1t_2t_3N$ is denoted by [1 12 1 2 3] and $Nt_1t_{12}t_1t_2t_3$ is its representative right coset.

$Nt_1t_{12}t_2t_1N$

Now $Nt_1t_{12}t_2t_1N$ in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(11221)} \geq \langle Id(N), (1,2,3)(4,9,11)(5,8,12)(6,7,10)\rangle$. Therefore the number of the single cosets in $Nt_1t_{12}t_2t_1N$ is $\frac{|N|}{|N^{(11221)}|} = \frac{24}{3} = 8$. The names of the single cosets are given below, let $\gamma = (1,2,3)(4,9,11)(5,8,12)(6,7,10)$
$(Nt_1t_{12}t_2t_1)^{\gamma} \sim Nt_2t_5t_3t_2 \Rightarrow (Nt_2t_5t_3t_2)^{\gamma} \sim Nt_3t_8t_1t_3$,
$(Nt_4t_{10}t_5t_4)^{\gamma} \sim Nt_9t_6t_8t_9 \Rightarrow (Nt_9t_6t_8t_9)^{\gamma} \sim Nt_{11}t_7t12t_{11}$,
$(Nt_7t_{11}t_8t_7)^{\gamma} \sim Nt_{10}t_4t12t_{10} \Rightarrow (Nt_{10}t_4t12t_{10})^{\gamma} \sim Nt_6t_9t_5t_6$,
$(Nt_1t_{12}t_3t_1)^{\gamma} \sim Nt_2t_5t_1t_2 \Rightarrow (Nt_2t_5t_1t_2)^{\gamma} \sim Nt_3t_8t_2t_3$,
$(Nt_7t_{11}t_9t_7)^{\gamma} \sim Nt_{10}t_4t_{11}t_{10} \Rightarrow (Nt_{10}t_4t_{11}t_{10})^{\gamma} \sim Nt_6t_9t_4t_6$,
$(Nt_4t_{10}t_6t_4)^{\gamma} \sim Nt_9t_6t_7t_9 \Rightarrow (Nt_9t_6t_7t_9)^{\gamma} \sim Nt_{11}t_7t_{10}t_{11}$,
$(Nt_5t_2t_6t_5)^{\gamma} \sim Nt_8t_3t_7t_8 \Rightarrow (Nt_8t_3t_7t_8)^{\gamma} \sim Nt_{12}t_1t_{10}t_{12}$,
$(Nt_8t_3t_9t_8)^{\gamma} \sim Nt_{12}t_1t11t_{12} \Rightarrow (Nt_{12}t_1t11t_{12})^{\gamma} \sim Nt_5t_2t_4t_5$
The orbits of $N^{(11221)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are:
$\mathbb{O} = \{1,2,3\}, \{4,9,11\}, \{5,8,12\}, \{6,7,10\}$. Take a representative t_i from each orbit and

see which double cosets $Nt_1t_{12}t_2t_1t_i$ belongs to. We have the following:

$$\begin{aligned} Nt_1t_{12}t_2t_1t_1 &= Nt_1t_{12}t_2 \in [1122] \\ t_1t_{12}t_2t_1t_4 &= yt_7t_{11}t_8 \\ \implies Nt_1t_{12}t_2t_1t_4 &= Nt_7t_{11}t_8 \in [1122] \\ t_1t_{12}t_2t_1t_5 &= xy^{-1}t_1t_{12}t_2t_1 \\ \implies Nt_1t_{12}t_2t_1t_5 &= Nt_1t_{12}t_2t_1 \in [11221] \\ t_1t_{12}t_2t_1t_6 &= t_{11}t_7t_{10} \\ \implies Nt_1t_{12}t_2t_1t_6 &= Nt_{11}t_7t_{10} \in [1122] \end{aligned}$$

$Nt_1t_{12}t_1t_2t_3N$

Now $Nt_1t_{12}t_1t_2t_3N$ in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(112123)} \geq \langle Id(N), (1,11,9,5)(2,12,7,6)(3,10,8,4)\rangle$ since $(Nt_1t_{12}t_1t_2t_3)^{(1,11,9,5)(2,12,7,6)(3,10,8,4)} = Nt_{11}t_7t_{11}t_12t_{10}$ where $(1,11,9,5)(2,12,7,6)$ $(3,10,8,4) \in N$. Then, $|N^{(112123)}| = 4$. So the number of single cosets in $N^{(112123)}$ is $\frac{|N|}{|N^{(112123)}|} = \frac{24}{4} = 6$ and the names are listed below. Let $\delta = (1,11,9,5)(2,12,7,6)$ $(3,10,8,4)$:

$(Nt_1t_{12}t_1t_2t_3)^\delta \sim Nt_{11}t_7t_{11}t_{12}t_{10} \Rightarrow (Nt_{11}t_7t_{11}t_{12}t_{10})^\delta \sim Nt_9t_6t_9t_7t_8 \Rightarrow (Nt_9t_6t_9t_7t_8)^\delta$
$\sim Nt_5t_2t_5t_6t_4$,
$(Nt_4t_{10}t_4t_5t_6)^\delta \sim Nt_3t_8t_3t_1t_2 \Rightarrow (Nt_3t_8t_3t_1t_2)^\delta \sim Nt_{10}t_4t_{10}t_{11}t_{12} \Rightarrow (Nt_{10}t_4t_{10}t_{11}t_{12})^\delta$
$\sim Nt_8t_3t_8t_9t_7$,
$(Nt_{11}t_7t_{11}t_{10}t_{12})^\delta \sim Nt_9t_6t_9t_8t_7 \Rightarrow (Nt_9t_6t_9t_8t_7)^\delta \sim Nt_5t_2t_5t_4t_6 \Rightarrow (Nt_5t_2t_5t_4t_6)^\delta$
$\sim Nt_1t_{12}t_1t_3t_2$,
$(Nt_8t_3t_8t_7t_9)^\delta \sim Nt_4t_{10}t_4t_6t_5 \Rightarrow (Nt_4t_{10}t_4t_6t_5)^\delta \sim Nt_3t_8t_3t_2t_1 \Rightarrow (Nt_3t_8t_3t_2t_1)^\delta$
$\sim Nt_{10}t_4t_{10}t_{12}t_{11}$,
$(Nt_2t_5t_2t_3t_1)^\delta \sim Nt_{12}t_1t_{12}t_{10}t_{11} \Rightarrow (Nt_{12}t_1t_{12}t_{10}t_{11})^\delta \sim Nt_7t_{11}t_7t_8t_9 \Rightarrow (Nt_7t_{11}t_7t_8t_9)^\delta$
$\sim Nt_6t_9t_6t_4t_5$,
$(Nt_7t_{11}t_7t_9t_8)^\delta \sim Nt_6t_9t_6t_5t_4 \Rightarrow (Nt_6t_9t_6t_5t_4)^\delta \sim Nt_2t_5t_2t_1t_3 \Rightarrow (Nt_2t_5t_2t_1t_3)^\delta$
$\sim Nt_{12}t_1t_{12}t_{11}t_{10}$

The orbits of $N^{(112123)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are:
$\mathbb{O} = \{1,11,9,5\},\{2,12,7,6\}$,

and$\{3, 10, 8, 4\}$. Take a representative t_i from each orbit and see which double coset $Nt_1t_{12}t_1t_2t_3t_i$ belongs to. We have

$$\begin{aligned} Nt_1t_{12}t_1t_2t_3t_1 &= yzt_9t_6t_9t_8 \\ \implies Nt_1t_{12}t_1t_2t_3t_1 &= Nt_9t_6t_9t_8 \in [11212] \\ t_1t_{12}t_1t_2t_3t_2 &= t_8t_3t_8t_7 \\ \implies Nt_1t_{12}t_1t_2t_3t_2 &= Nt_8t_3t_8t_7 \in [11212] \\ Nt_1t_{12}t_1t_2t_3t_3 &= Nt_1t_{12}t_1t_2 \in [11212] \end{aligned}$$

After multiplying on the right by an element from each orbit we have concluded that there are no new double cosets. Then we checked and prove for those double cosets if they are equal to other existing double cosets.

We have completed the double coset enumeration of G, since the set of right cosets is closed under multiplication. Thus the index of N in G is 2184. We have concluded the following:

$$G = NeN \cup Nt_1N \cup Nt_1t_{12}N \cup Nt_1t_{12}t_1N \cup Nt_1t_{12}t_2N \cup Nt_1t_{12}t_1t_2N$$

$$\cup Nt_1t_{12}t_2t_1N \cup Nt_1t_{12}t_1t_2t_3N$$

where

$$G \cong \frac{2^{*12} : S_4}{(x * y^{-1} * z^{y^{-2}} * t)^7, (y * t)^2} \cong PGL_2(13).$$

Therefore,

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(112)}|} + \frac{|N|}{|N^{(1121)}|} + \frac{|N|}{|N^{(1122)}|} + \frac{|N|}{|N^{(11212)}|} + \frac{|N|}{|N^{(11221)}|} + \frac{|N|}{|N^{(112123)}|}$$

and

$|G| \leq (1 + 4 + 12 + 12 + 24 + 24 + 8 + 6) \times 24$

$|G| \leq 2184.$

The Cayley graph summarizes the information listed above.

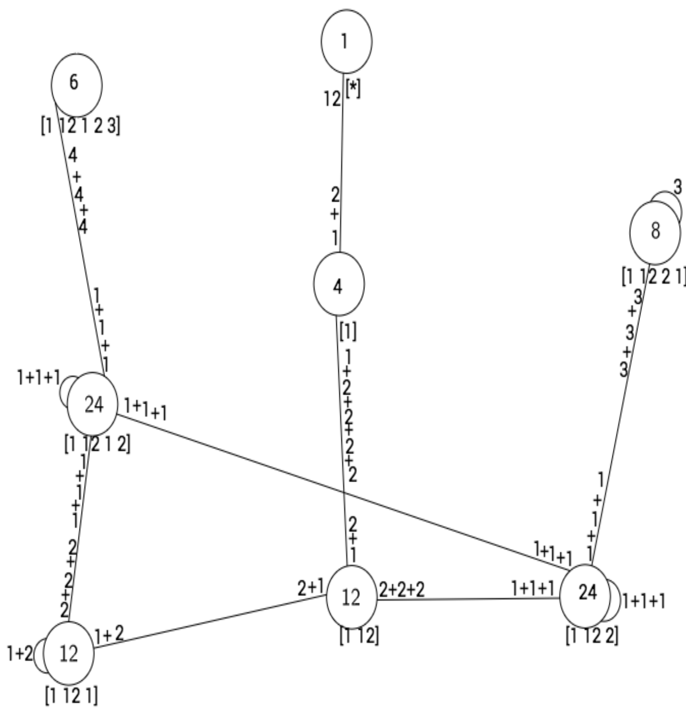

Figure 4.2: Cayley graph of $PGL_2(13)$ over S_4

4.3.2 Permutation Representation of $PGL_2(13)$ over S_4

Next we compute the action of x, y, and z on the 91 single cosets of N in G. We also right multiply $t \sim t_1$ to all the 91 single cosets. The work is organized on the following table.

Table 4.2: Permutation Representation of $PGL_2(13)$ over S_4

Cosets	$x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$	$y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$	$z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$	$t \sim t_1$
1. N	1. N	1. N	1. N	2. Nt_1
2. Nt_7	2. Nt_7	2. Nt_7	3. Nt_{11}	1. $Nt_7t_1 = N$
3. Nt_{11}	3. Nt_{11}	4. Nt_{10}	2. Nt_7	5. $Nt_{11}t_1$
4. Nt_{10}	6. Nt_{12}	6. Nt_{12}	6. Nt_{12}	7. $Nt_{10}t_1$
5. $Nt_{10}t_4$	8. $Nt_{12}t_1$	8. $Nt_{12}t_1$	9. Nt_8t_3	3. $Nt_{10}t_4t_1$
6. Nt_{12}	4. Nt_{10}	3. Nt_{11}	4. Nt_{10}	8. $Nt_{12}t_1$
7. $Nt_{11}t_7$	7. $Nt_{11}t_7$	5. $Nt_{10}t_4$	10. Nt_7t_{11}	4. Nt_{10}
8. $Nt_{12}t_1$	5. $Nt_{10}t_4$	7. $Nt_{11}t_7$	11. Nt_9t_6	6. $Nt_{12}t_1t_1 = Nt_{12}$
9. Nt_8t_3	11. Nt_9t_6	12. Nt_6t_9	5. $Nt_{10}t_4$	13. $Nt_6t_9t_4$
10. Nt_7t_{11}	10. Nt_7t_{11}	15. Nt_4t_{10}	7. $Nt_{11}t_7$	16. $Nt_4t_{10}t_4$
11. Nt_9t_6	9. Nt_8t_3	18. Nt_5t_2	8. $Nt_{12}t_1$	19. $Nt_5t_2t_4$
12. Nt_6t_9	20. Nt_3t_8	14. Nt_2t_5	17. Nt_1t_{12}	21. $Nt_8t_3t_7$
13. $Nt_6t_9t_4$	22. $Nt_8t_3t_1$	23. $Nt_2t_5t_1$	24. $Nt_1t_{12}t_3$	9. Nt_8t_3
14. Nt_2t_5	18. Nt_5t_2	9. Nt_8t_3	18. Nt_5t_2	23. $Nt_2t_5t_1$
15. Nt_4t_{10}	17. Nt_1t_{12}	17. Nt_1t_{12}	20. Nt_3t_8	25. $Nt_7t_{11}t_7$
16. $Nt_4t_{10}t_4$	26. $Nt_1t_{12}t_1$	26. $Nt_1t_{12}t_1$	27. $Nt_3t_8t_3$	10. Nt_7t_{11}
17. Nt_1t_{12}	15. Nt_4t_{10}	10. Nt_7t_{11}	12. Nt_6t_9	26. $Nt_1t_{12}t_1$
18. Nt_5t_2	14. Nt_2t_5	20. Nt_3t_8	14. Nt_2t_5	28. $Nt_9t_6t_7$
19. $Nt_5t_2t_4$	23. $Nt_2t_5t_1$	22. $Nt_3t_8t_1$	29. $Nt_2t_5t_3$	11. Nt_9t_6
20. Nt_3t_8	12. Nt_6t_9	11. Nt_9t_6	15. Nt_4t_{10}	22. $Nt_3t_8t_1$
21. $Nt_8t_3t_7$	21. $Nt_9t_6t_7$	13. $Nt_6t_9t_4$	30. $Nt_{10}t_4t_{11}$	12. Nt_6t_9
22. $Nt_3t_8t_1$	13. $Nt_6t_9t_4$	28. $Nt_9t_6t_7$	31. $Nt_4t_{10}t_6$	20. Nt_3t_8
23. $Nt_2t_5t_1$	19. $Nt_5t_2t_4$	21. $Nt_8t_3t_7$	32. $Nt_5t_2t_6$	14. Nt_2t_5
24. $Nt_1t_{12}t_3$	31. $Nt_4t_{10}t_6$	33. $Nt_7t_{11}t_9$	13. $Nt_6t_9t_4$	34. $Nt_1t_{12}t_3t_1$
25. $Nt_7t_{11}t_7$	25. $Nt_7t_{11}t_7$	16. $Nt_4t_{10}t_4$	36. $Nt_{11}t_7t_{11}$	15. Nt_4t_{10}
26. $Nt_1t_{12}t_1$	16. $Nt_4t_{10}t_4$	25. $Nt_7t_{11}t_7$	37. $Nt_6t_9t_6$	17. Nt_1t_{12}
27. $Nt_3t_8t_3$	37. $Nt_6t_9t_6$	38. $Nt_9t_6t_9$	16. $Nt_4t_{10}t_4$	39. $Nt_3t_8t_3t_1$
28. $Nt_9t_6t_7$	21. $Nt_8t_3t_7$	19. $Nt_5t_2t_4$	41. $Nt_{12}t_1t_{11}$	18. Nt_5t_2
29. $Nt_2t_5t_3$	32. $Nt_5t_2t_6$	42. $Nt_8t_3t_9$	19. $Nt_5t_2t_4$	43. $Nt_6t_9t_5$
30. $Nt_{10}t_4t_{11}$	41. $Nt12t1t11$	44. $Nt_{12}t_1t_{10}$	21. $Nt_8t_3t_7$	45. $Nt_4t_{10}t_4t_6$
31. $Nt_4t_{10}t_6$	24. $Nt_1t_{12}t_3$	47. $Nt_1t_{12}t_2$	22. $Nt_3t_8t_1$	48. $Nt_7t_{11}t_8t_7$
32. $Nt_5t_2t_6$	29. $Nt_2t_5t_3$	50. $Nt_3t_8t_2$	23. $Nt_2t_5t_1$	32. $Nt_5t_2t_6$
33. $Nt_7t_{11}t_9$	49. $Nt_7t_{11}t_8$	35. $Nt_4t_{10}t_5$	46. $Nt_{11}t_7t_{12}$	52. $Nt_4t_{10}t_5t_4$
34. $Nt_1t_{12}t_3t_1$	53. $Nt_4t_{10}t_6t_4$	54. $Nt_7t_{11}t_9t_7$	52. $Nt_4t_{10}t_5t_4$	24. $Nt_1t_{12}t_3$
35. $Nt_4t_{10}t_5$	47. $Nt_1t_{12}t_2$	24. $Nt_1t_{12}t_3$	50. $Nt_3t_8t_2$	54. $Nt_7t_{11}t_9t_7$
36. $Nt_{11}t_7t_{11}$	36. $Nt_{11}t_7t_{11}$	55. $Nt_{10}t_4t_{10}$	25. $Nt_7t_{11}t_7$	55. $Nt_{10}t_4t_{10}$
37. $Nt_6t_9t_6$	27. $Nt_3t_8t_3$	57. $Nt_2t_5t_2$	26. $Nt_1t_{12}t_1$	58. $Nt_8t_3t_8t_7$
38. $Nt_9t_6t_9$	59. $Nt_8t_3t_8$	40. $Nt_5t_2t_5$	56. $Nt_{12}t_1t_{12}$	60. $Nt_5t_2t_5t_4$
39. $Nt_3t_8t_3t_1$	61. $Nt_6t_9t_6t_4$	62. $Nt_9t_6t_9t_7$	45. $Nt_4t_{10}t_4t_6$	27. $Nt_3t_8t_3$
40. $Nt_5t_2t_5$	57. $Nt_2t_5t_2$	27. $Nt_3t_8t_3$	57. $Nt_2t_5t_2$	62. $Nt_9t_6t_9t_7$
41. $Nt_{12}t_1t_{11}$	30. $Nt_{10}t_4t11$	63. $Nt_{11}t_7t_{10}$	28. $Nt_9t_6t_7$	64. $Nt_4t_{10}t_4t_5$
42. $Nt_8t_3t_9$	51. $Nt_9t_6t_8$	43. $Nt_6t_9t_5$	65. $Nt_{10}t_4t_{12}$	42. $t_8t_3t_9$
43. $Nt_6t_9t_5$	50. $Nt_3t_8t_2$	29. $Nt_2t_5t_3$	47. $Nt_1t_{12}t_2$	29. $Nt_2t_5t_3$
44. $Nt_{12}t_1t_{10}$	65. $Nt_{10}t_4t_{12}$	46. $Nt_{11}t_7t_{12}$	51. $Nt_9t_6t_8$	66. $Nt_7t_{11}t_7t_8$
45. $Nt_4t_{10}t_4t_6$	67. $Nt_1t_{12}t_1t_3$	68. $Nt_1t_{12}t_1t_2$	39. $Nt_3t_8t_3t_1$	30. $Nt_{10}t_4t_{11}$
46. $Nt_{11}t_7t_{12}$	63. $Nt_{11}t_7t_{10}$	30. $Nt_{10}t_4t_{11}$	33. $Nt_7t_{11}t_9$	68. $Nt_1t_{12}t_1t_2$
47. $Nt_1t_{12}t_2$	35. $Nt_4t_{10}t_5$	49. $Nt_7t_{11}t_8$	43. $Nt_6t_9t_5$	69. $Nt_1t_{12}t_2t_1$
48. $Nt_7t_{11}t_8t_7$	54. $Nt_7t_{11}t_9t_7$	53. $Nt_4t_{10}t_6t_4$	70. $Nt_{11}t_7t_{10}t_{11}$	31. $Nt_4t_{10}t_6$
49. $Nt_7t_{11}t_8$	33. $Nt_7t_{11}t_9$	31. $Nt_4t_{10}t_6$	63. $Nt_{11}t_7t_{10}$	53. $Nt_4t_{10}t_6t_4$
50. $Nt_3t_8t_2$	43. $Nt_6t_9t_5$	51. $Nt_9t_6t_8$	35. $Nt_4t_{10}t_5$	51. $Nt_9t_6t_8$
51. $Nt_9t_6t_8$	42. $Nt_8t_3t_9$	32. $Nt_5t_2t_6$	44. $Nt_{12}t_1t_{10}$	50. $Nt_3t_8t_2$
52. $Nt_4t_{10}t_5t_4$	69. $Nt_1t_{12}t_2t_1$	34. $Nt_1t_{12}t_3t_1$	34. $Nt_1t_{12}t_3t_1$	33. $Nt_7t_{11}t_9$
53. $Nt_4t_{10}t_6t_4$	34. $Nt_1t_{12}t_3t_1$	69. $Nt_1t_{12}t_2t_1$	69. $Nt_1t_{12}t_2t_1$	49. $Nt_7t_{11}t_8$
54. $Nt_7t_{11}t_9t_7$	48. $Nt_7t_{11}t_8t_7$	52. $Nt_4t_{10}t_5t_4$	71. $Nt_{11}t_7t_{12}t_{11}$	35. $Nt_4t_{10}t_5$
55. $Nt_{10}t_4t_{10}$	56. $Nt_{12}t_1t_{12}$	56. $Nt_{12}t_1t_{12}$	59. $Nt_8t_3t_8$	36. $Nt_{11}t_7t_{11}$
56. $Nt_{12}t_1t_{12}$	55. $Nt_{10}t_4t_{10}$	36. $Nt_{11}t_7t_{11}$	38. $Nt_9t_6t_9$	56. $Nt_{12}t_1t_{12}$
57. $Nt_2t_5t_2$	40. $Nt_5t_2t_5$	59. $Nt_8t_3t_8$	40. $Nt_5t_2t_5$	72. $Nt_2t_5t_2t_1$
58. $Nt_8t_3t_8t_7$	62. $Nt_9t_6t_9$	61. $Nt_6t_9t_6t_4$	73. $Nt_{10}t_4t_{10}t_{11}$	37. $Nt_6t_9t_6$
59. $Nt_8t_3t_8$	38. $Nt_9t_6t_9$	37. $Nt_6t_9t_6$	55. $Nt_{10}t_4t_{10}$	61. $Nt_6t_9t_6t_4$
60. $Nt_5t_2t_5t_4$	72. $Nt_2t_5t_2t_1$	39. $Nt_3t_8t_3t_1$	74. $Nt_2t_5t_2t_3$	38. $Nt_9t_6t_9$

Table 4.3: Permutation Representation of $PGL_2(13)$ over S_4 Cont.

Cosets	$x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$	$y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$	$z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$	$t \sim t_1$
61. $Nt_6t_9t_6t_4$	39. $Nt_3t_8t_3t_1$	72. $Nt_2t_5t_2t_1$	67. $Nt_1t_{12}t_1t_3$	59. $Nt_8t_3t_8$
62. $Nt_9t_6t_9t_7$	58. $Nt_8t_3t_8t_7$	60. $Nt_5t_2t_5t_4$	75. $Nt_{12}t_1t_{12}t_{11}$	40. $Nt_5t_2t_5$
63. $Nt_{11}t_7t_{10}$	46. $Nt_{11}t_7t12$	65. $Nt_{10}t_4t_{12}$	49. $Nt_7t_{11}t_8$	76. $Nt_7t_{11}t_7t_9$
64. $Nt_4t_{10}t_4t_5$	68. $Nt_1t_{12}t_1t_2$	67. $Nt_1t_{12}t_1t_3$	77. $Nt_3t_8t_3t_2$	41. $Nt_{12}t_1t_{11}$
65. $Nt_{10}t_4t_{12}$	44. $Nt_{12}t_1t_{10}$	41. $Nt_{12}t_1t_{11}$	42. $Nt_8t_3t_9$	67. $Nt_1t_{12}t_1t_3$
66. $Nt_7t_{11}t_7t_8$	76. $Nt_7t_{11}t_7t_9$	45. $Nt_4t_{10}t_4t_6$	78. $Nt_{11}t_7t_{11}t_{10}$	44. $Nt_{12}t_1t_{10}$
67. $Nt_1t_{12}t_1t_3$	45. $Nt_4t_{10}t_4t_6$	76. $Nt_7t_{11}t_7t_9$	61. $Nt_6t_9t_6t_4$	65. $Nt_{10}t_4t_{12}$
68. $Nt_1t_{12}t_1t_2$	64. $Nt_4t_{10}t_4t_5$	66. $Nt_7t_{11}t_7t_8$	79. $Nt_6t_9t_6t_5$	46. $Nt_{11}t_7t_{12}$
69. $Nt_1t_{12}t_2t_1$	52. $Nt_4t_{10}t_5t_4$	48. $Nt_7t_{11}t_8t_7$	53. $Nt_4t_{10}t_6t_4$	47. $Nt_1t_{12}t_2$
70. $Nt_{11}t_7t_{10}t_{11}$	71. $Nt_{11}t_7t_{12}t_{11}$	70. $Nt_{11}t_7t_{10}t_{11}$	48. $Nt_7t_{11}t_8t_7$	70. $Nt_{11}t_7t_{10}t_{11}$
71. $Nt_{11}t_7t_{12}t_{11}$	70. $Nt_{11}t_7t_{10}t_{11}$	71. $Nt_{11}t_7t_{12}t_{11}$	54. $Nt_7t_{11}t_9t_7$	71. $Nt_{11}t_7t_{12}t_{11}$
72. $Nt_2t_5t_2t_1$	60. $Nt_5t_2t_5t_4$	58. $Nt_8t_3t_8t_7$	80. $Nt_5t_2t_5t_6$	57. $Nt_2t_5t_2$
73. $Nt_{10}t_4t_{10}t_{11}$	75. $Nt_{12}t_1t_{12}t_{11}$	81. $Nt_{12}t_1t_{12}t_{10}$	58. $Nt_8t_3t_8t_7$	82. $Nt_{11}t_7t_{11}t_{12}$
74. $Nt_2t_5t_2t_3$	80. $Nt_5t_2t_5t_6$	83. $Nt_8t_3t_8t_9$	60. $Nt_5t_2t_5t_4$	84. $Nt_4t_{10}t_4t_5t_6$
75. $Nt_{12}t_1t_{12}t_{11}$	73. $Nt_{10}t_4t_{10}t_{11}$	78. $Nt_{11}t_7t_{11}t_{10}$	62. $Nt_9t_6t_9t_7$	75. $Nt_{12}t_1t_{12}t_{11}$
76. $Nt_7t_{11}t_7t_9$	66. $Nt_7t_{11}t_7t_8$	64. $Nt_4t_{10}t_4t_5$	82. $Nt_{11}t_7t_{11}t_{12}$	63. $Nt_{11}t_7t_{10}$
77. $Nt_3t_8t_3t_2$	79. $Nt_6t_9t_6t_5$	86. $Nt_9t_6t_9t_8$	64. $Nt_4t_{10}t_4t_5$	87. $Nt_7t_{11}t_7t_8t_9$
78. $Nt_{11}t_7t_{11}t_{10}$	82. $Nt_{11}t_7t_{11}t_{12}$	85. $Nt_{10}t_4t_{10}t_{12}$	66. $Nt_7t_{11}t_7t_8$	85. $Nt_{10}t_4t_{10}t_{12}$
79. $Nt_6t_9t_6t_5$	77. $Nt_3t_8t_3t_2$	74. $Nt_2t_5t_2t_3$	68. $Nt_1t_{12}t_1t_2$	88. $Nt_1t_{12}t_1t_3t_2$
80. $Nt_5t_2t_5t_6$	74. $Nt_2t_5t_2t_3$	77. $Nt_3t_8t_3t_2$	72. $Nt_2t_5t_2t_1$	89. $Nt_4t_{10}t_4t_6t_5$
81. $Nt_{12}t_1t_{12}t_{10}$	85. $Nt_{10}t_4t_{10}t_{12}$	82. $Nt_{11}t_7t_{11}t_{12}$	86. $Nt_9t_6t_9t_8$	81. $Nt_{12}t_1t_{12}t_{10}$
82. $Nt_{11}t_7t_{11}t_{12}$	78. $Nt_{11}t_7t_{11}t_{10}$	73. $Nt_{10}t_4t_{10}t_{11}$	76. $Nt_7t_{11}t_7t_9$	73. $Nt_{10}t_4t_{10}t_{11}$
83. $Nt_8t_3t_8t_9$	86. $Nt_9t_6t_9t_8$	79. $Nt_6t_9t_6t_5$	85. $Nt_{10}t_4t_{10}t_{12}$	90. $Nt_7t_{11}t_7t_9t_8$
84. $Nt_4t_{10}t_4t_5t_6$	91. $Nt_1t_{12}t_1t_2t_3$	88. $Nt_1t_{12}t_1t_3t_2$	87. $Nt_7t_{11}t_7t_8t_9$	74. $Nt_2t_5t_2t_3$
85. $Nt_{10}t_4t_{10}t_{12}$	81. $Nt_{12}t_1t_{12}t_{10}$	75. $Nt_{12}t_1t_{12}t_{11}$	83. $Nt_8t_3t_8t_9$	78. $Nt_{11}t_7t_{11}t_{10}$
86. $Nt_9t_6t_9t_8$	83. $Nt_8t_3t_8t_9$	80. $Nt_5t_2t_5t_6$	81. $Nt_{12}t_1t_{12}t_{10}$	91. $Nt_1t_{12}t_1t_2t_3$
87. $Nt_7t_{11}t_7t_8t_9$	90. $Nt_7t_{11}t_7t_9t_8$	89. $Nt_4t_{10}t_4t_6t_5$	84. $Nt_4t_{10}t_4t_5t_6$	77. $Nt_3t_8t_3t_2$
88. $Nt_1t_{12}t_1t_3t_2$	89. $Nt_4t_{10}t_4t_6t_5$	90. $Nt_7t_{11}t_7t_9t_8$	88. $Nt_1t_{12}t_1t_3t_2$	79. $Nt_6t_9t_6t_5$
89. $Nt_4t_{10}t_4t_6t_5$	88. $Nt_1t_{12}t_1t_3t_2$	91. $Nt_1t_{12}t_1t_2t_3$	89. $Nt_4t_{10}t_4t_6t_5$	80. $Nt_5t_2t_5t_6$
90. $Nt_7t_{11}t_7t_9t_8$	87. $Nt_7t_{11}t_7t_8t_9$	84. $Nt_4t_{10}t_4t_5t_6$	91. $Nt_1t_{12}t_1t_2t_3$	83. $Nt_8t_3t_8t_9$
91. $Nt_1t_{12}t_1t_2t_3$	84. $Nt_4t_{10}t_4t_5t_6$	87. $Nt_7t_{11}t_7t_8t_9$	90. $Nt_7t_{11}t_7t_9t_8$	86. $Nt_9t_6t_9t_8$

We have the following:

$\phi(x) = (4,6)(5,8)(9,11)(12,20)(13,22)(14,18)(15,17)(16,26)$
$(19,23)(21,28)(24,31)(27,37)(29,32)(30,41)(33,49)(34,53)(35,$
$47)(38,59)(39,61)(40,57)(42,51)(43,50)(44,65)(45,67)(46,63)$
$(48,54)(52,69)(55,56)(58,62)(60,72)(64,68)(66,76)(70,71)(73,$
$75)(74,80)(77,79)(78,82)(81,85)(83,86)(84,91)(87,90)(88,89)$

$\phi(y) = (3,4,6)(5,8,7)(9,12,14)(10,15,17)(11,18,20)(13,23,21)$
$(16,26,25)(19,22,28)(24,33,35)(27,38,40)(29,42,43)(30,44,46)$
$(31,47,49)(32,50,51)(34,54,52)(36,55,56)(37,57,59)(39,62,60)$
$(41,63,65)(45,68,66)(48,53,69)(58,61,72)(64,67,76)(73,81,82)$
$(74,83,79)(75,78,85)(77,86,80)(84,88,90)(87,89,91)$

$\phi(z) = (2,3)(4,6)(5,9)(7,10)(8,11)(12,17)(13,24)(14,18)(15,20)$

$(16,27)(19,29)(21,30)(22,31)(23,32)(25,36)(26,37)(28,41)(33,46)$
$(34,52)(35,50)(38,56)(39,45)(40,57)(42,65)(43,47)(44,51)(48,70)$
$(49,63)(53,69)(54,71)(55,59)(58,73)(60,74)(61,67)(62,75)(64,77)$
$(66,78)(68,79)(72,80)(76,82)(81,86)(83,85)(84,87)(90,91)$

$\phi(t) = (1,2)(3,5)(4,7)(6,8)(9,13)(10,16)(11,19)(12,21)(14,23)$
$(15,25)(17,26)(18,28)(20,22)(24,34)(27,39)(29,43)(30,45)(31,48)$
$(33,52)(35,54)(36,55)(37,58)(38,60)(40,62)(41,64)(44,66)(46,68)$
$(47,69)(49,53)(50,51)(57,72)(59,61)(63,76)(65,67)(73,82)(74,84)$
$(77,87)(78,85)(79,88)(80,89)(83,90)(86,91)$

Hence, we have $\phi : 2^{*12} : S_4 \longrightarrow S_{91}$ since G acts on $X = \{N, Nt_1, Nt_{12}, Nt_1t_{12}t_1, Nt_1t_{12}t_2, Nt_1t_{12}t_1t_2N, Nt_1t_{12}t_2t_1, Nt_1t_{12}t_1t_2t_3\}$ with $|X| = 91$. Then $\phi(G) =< \phi(x), \phi(y), \phi(z), \phi(t) >$. To prove that $\phi(G) =< \phi(x), \phi(y), \phi(z), \phi(t) >$ is a homomorphic image of $G = 2^{*12} : S_4$, the following conditions must be satisfied:

1. $\phi(N) \cong S_4$
2. $\phi(t)$has twelve conjugates under conjugation by $\phi(N)$
3. $\phi(N)$ acts as S_4 on the twelve conjugates of $\phi(t)$ by conjugates

Proof. Let $\phi(N) =< \phi(x), \phi(y), \phi(z) >$.

1. We want to show $\phi(N) =< \phi(x), \phi(y), \phi(z) > \cong S_4$.
   Let the presentation of S_4 be
   $x^2, y^3, z^2, (y^{-1}x)^2, y^{-1}zy(z^{(y^{-2})}), (xz)^2, (z(z^{(y^{-2})}))^2, yzy^{-1}z(z^{(y^{-2})})$.
   Now we check if the presentation holds for $\phi(N)$.
   $|\phi(x)| = |(4,6)(5,8)(9,11)...| = 2, |\phi(y)| = |(3,4,6)(5,8,7)...| = 3,$
   $|\phi(z)| = 2,\ |(\phi(y^{-1})\phi(x))| = |(3,4)(5,7)(9,18)...| = 2,$
   $|(\phi(x)\phi(z))| = |(2,3)(5,11)(7,10)...| = 2,\ |\phi(y^{-1})\phi(z)\phi(y)(\phi(z)^{(\phi(y^{-2}))})| = 1$, and
   $|\phi(y)\phi(z)\phi(y^{-1})\phi(z)(\phi(z)^{\phi(y^{-2})})| = 1.$
   Therefore, $\phi(N) \cong S_4$.

2. We compute $\phi(t)^{\phi(N)}$:

$\phi(t)^{\phi(y^3)} = \phi(t)^e = \{(1,2)(3,5)(4,7)(6,8)(9,13)...\} = t_1$

$\phi(t)^{\phi(z)\phi(y)} = \{(1,4)(2,12)(3,15)...\} = t_2$

$\phi(t)^{\phi(x)^{\phi(z)^{\phi(y^{-2})}}} = \{(1,2)(3,5)...^{(2,3)(5,11)(7,10)...}\}$

$= \{(1,3)(2,11)(4,10)...\} = t_3$

$\phi(t)^{\phi(x)} = \{(1,2)(3,8)(4,5)...^{(4,6)(5,8)(9,11)...}\}$

$= \{(1,2)(3,8)(4,5)...\} = t_4$

$\phi(t)^{(\phi(z)^{\phi(y)^{-2}}\phi(y))} = \{(1,2)(3,5)(4,7)...^{(2,6,4)(5,17,18)...}\}$

$= \{(1,6)(2,20)(3,17)...\} = t_5$

$\phi(t)^{\phi(z)} = \{(1,2)(3,5)(4,7)...^{(2,3)(4,6)(5,9)...}\}$

$= \{(1,3)(2,9)(4,11)...\} = t_6$

$\phi(t)^{\phi(y)} = \{(1,2)(3,5)(4,7)...^{(3,4,6)(5,8,7)...}\}$

$= \{(1,2)(3,7)(4,8))...\} = t_7$

$\phi(t)^{(\phi(z)^{\phi(y)^{-2}}\phi(x))} = \{(1,2)(3,5)(4,7)...^{(2,6,3,4)(5,17,9,12)...}\}$

$= \{(1,6)(2,14)(3,20)...\} = t_8$

$\phi(t)^{(\phi(z)^{\phi(y)^{-2}})} = \{(1,2)(3,5)(4,7)...^{(2,4)(3,6)(5,15)...}\}$

$= \{(1,4)(2,18)(3,12)...\} = t_9$

$\phi(t)^{\phi(x)(\phi(z)^{\phi(y^{-2})})^{-1}} = \{(1,2)(3,5)(4,7)...^{(2,4,3,6)(5,12,9,17)...}\}$

$= \{(1,4)(2,15)(3,18)...\} = t_{10}$

$\phi(t)^{(\phi(z)\phi(x)\phi(y^{-1}))^3} = \{(1,2)(3,5)(4,7)...^{(2,3,4,6)(5,9,15,20)...}\}$

$= \{(1,3)(2,10)(4,9)(5,30)...\} = t_{11}$

$\phi(t)^{\phi(z)(\phi(z)^{\phi(y^{-2})})} = \{(1,2)(3,5)(4,7)...^{(2,6)(3,4)(5,20)..}\}$

$= \{(1,6)(2,17)(3,14)(4,20)...\} = t_{12}$

Therefore, $\phi(t)^{\phi(N)} = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}\}$.

3. Now we need to show that $\phi(N)$ acts as S_4 on the twelve conjugates of $\phi(t)$ by conjugates.

First, we conjugate by $\phi(x)$.

$t_1^{\phi(x)} = (1,2)(3,8)(4,5)... = t_4, \quad t_4^{\phi(x)} = (1,2)(3,5)(4,7)... = t_1,$

$t_2^{\phi(x)} = (1,6)(2,20)(3,17)... = t_5, \quad t_5^{\phi(x)} = (1,4)(2,12)(3,15)... = t_2,$

$t_3^{\phi(x)} = (1,3)(2,9)(4,11)... = t_6, \quad t_6^{\phi(x)} = (1,3)(2,11)(4,10)... = t_3,$

$t_8^{\phi(x)} = (1,4)(2,18)(3,12)... = t_9, \quad t_9^{\phi(x)} = (1,6)(2,14)(3,20)... = t_8,$
$t_{10}^{\phi(x)} = (1,6)(2,17)(3,14)... = t_{12}, \quad t_{12}^{\phi(x)} = (1,4)(2,15)(3,18)... = t_{10}$
Thus, $\phi(x) = (t_1, t_4)(t_2, t_5)(t_3, t_6)(t_8, t_9)(t_{10}, t_{12})$.

Next, we conjugate by $\phi(y)$.
$t_1^{\phi(y)} = (1,2)(3,7)(4,8)... = t_7, \quad t_7^{\phi(y)} = (1,2)(3,8)(4,5)... = t_4,$
$t_4^{\phi(y)} = (1,2)(3,5)(4,7)... = t_1,$
$t_2^{\phi(y)} = (1,6)(2,14)(3,20)... = t_8, \quad t_8^{\phi(y)} = (1,3)(2,9)(4,11)... = t_6,$
$t_6^{\phi(y)} = (1,4)(2,12)(3,15)... = t_2,$
$t_3^{\phi(y)} = (1,4)(2,18)(3,12)... = t_9, \quad t_9^{\phi(y)} = (1,6)(2,20)(3,17)... = t_5,$
$t_5^{\phi(y)} = (1,3)(2,11)(4,10)... = t_3,$
$t_{10}^{\phi(y)} = (1,6)(2,17)(3,14)... = t_{12}, \quad t_{12}^{\phi(y)} = (1,3)(2,10)(4,9)... = t_{11},$
$t_{11}^{\phi(y)} = (1,4)(2,15)(3,18)... = t_{10}.$
Hence, $\phi(y) = (t_1, t_7, t_4)(t_2, t_8, t_6)(t_3, t_9, t_5)(t_{10}, t_{12}, t_{11})$.

Lastly, we conjugate by $\phi(z)$.
$t_1^{\phi(z)} = (1,3)(2,9)(4,11)... = t_6, \quad t_6^{\phi(z)} = (1,2)(3,5)(4,7)... = t_1,$
$t_2^{\phi(z)} = (1,6)(2,20)(3,17)... = t_5, \quad t_5^{\phi(z)} = (1,4)(2,12)(3,15)... = t_2,$
$t_3^{\phi(z)} = (1,2)(3,8)(4,5)... = t_4, \quad t_4^{\phi(z)} = (1,3)(2,11)(4,10)... = t_3,$
$t_7^{\phi(z)} = (1,3)(2,10)(4,9)... = t_{11}, \quad t_{11}^{\phi(z)} = (1,2)(3,7)(4,8)... = t_7,$
$t_8^{\phi(z)} = (1,4)(2,15)(3,18)... = t_{10}, \quad t_{10}^{\phi(z)} = (1,6)(2,14)(3,20)... = t_8$
$t_9^{\phi(z)} = (1,6)(2,17)(3,14)... = t_{12}, \quad t_{12}^{\phi(z)} = (1,4)(2,18)(3,12)... = t_9$
Thus, $\phi(z) = (t_1, t_6)(t_2, t_5)(t_3, t_4)(t_7, t_{11})(t_8, t_{10})(t_9, t_{12})$.
Therefore, $\phi(G) = \langle \phi(x), \phi(y), \phi(z), \phi(t) \rangle$ is a homomorphic image of $2^{*12} : S_4$.
By the First Isomorphism Theorem we have:

$$G/Ker\phi \cong \phi(G)$$

$$\implies |G/Ker\phi| \cong |\phi(G)|$$

$$\implies |G| = |Ker\phi||\phi(G)|.$$

If we look back to the double coset enumeration we know $|G| \leq 2184$. Furthermore, by MAGMA we were able to calculate $|\phi(G)|$.

$$|\phi(G)| = |\langle \phi(x), \phi(y), \phi(z), \phi(t) \rangle| = 2184.$$

$$|G| = |Ker\phi| 2184$$

$$\implies |G| \geq 2184.$$

So, $|Ker\phi| = 1$.

Therefore, $|G| = 2184$.

Chapter 5

Double Coset Enumeration of Simple(Twisted) Groups

5.1 Construction of $2^{\bullet}Sz(8)$ over $(13:4)$

5.1.1 Double Coset Enumeration of G

Consider the group $G = 2^{*13} : D_{14}$ factored by the relators $(y^{-1}x^{-1}t_1^{x^6})^5$ and $(x^2t_1)^5$ with symmetric presentation given by

$$\langle x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^{-2},$$

$$t^2,$$

$$(t, y^x), ((y^{-1}x^{-1})t^{(x^6)})^5, (x^2t)^5\rangle$$

where $N \cong (13:4) =$

$$\langle x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^{-2}\rangle.$$

Then,

$$G \cong \frac{2^{*13} : (13:4)}{((y^{-1}x^{-1}t_1^{x^6})^5, (x^2t_1)^5,} \cong 2^{\bullet}S_z(8).$$

Note $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$,

$y \sim (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7)$ and $t \sim t_1$. Let us expand the relations:

Assume $\pi = y^{-1}x^{-1} = (1,4,6,3)(2,9,5,11)(7,8,13,12)$ and $x^6 = (1,7,13,6,12,5,11,4,10,3,9,2,8)$

$$\left(\pi t^{x^6}\right)^5 = \left(\pi t_1^{(1,7,13,6,12,5,11,4,10,3,9,2,8)}\right)^5 = 1$$

$$(\pi t_7)^5 = 1$$

$$\implies (\pi t_7)^5 = \pi^5 t_7^{\pi^4} t_7^{\pi^3} t_7^{\pi^2} t_7^{\pi} t_7 = 1$$

$$\implies \pi^5 t_7^{\pi^4} t_7^{\pi^3} t_7^{\pi^2} t_7^{\pi} t_7 = 1$$

$$(1,4,6,3)(2,9,5,11)(7,8,13,12)t_7t_{12}t_{13}t_8t_7 = 1$$

$$\implies t_7t_8 = (1,4,6,3)(2,9,5,11)(7,8,13,12)t_7t_{12}t_{13}$$

and

Let $\beta = x^2 = (1,3,5,7,9,11,13,2,4,6,8,10,12)$

$$\implies (\beta t)^5 = \beta^5 t_1^{\beta^4} t_1^{\beta^3} t_1^{\beta^2} t_1^{\beta} t_1 = 1$$

$$\beta^5 t_1^{\beta^4} t_1^{\beta^3} t_1^{\beta^2} t_1^{\beta} t_1 = 1$$

$$\implies (1,11,8,5,2,12,9,6,3,13,10,7,4)t_9t_7t_5t_3t_1 = 1$$

$$\implies t_1t_3 = (1,11,8,5,2,12,9,6,3,13,10,7,4)t_9t_7t_5$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$ where g_i's words in t_i's.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [Nw^n | n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by t_i's. We need to identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilising group $N^{(w)}$

NeN

First double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$, the orbit of N on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ is: $\mathbb{O} = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$. Let t_1 be our symmetric generator from this orbit $\mathbb{O}$ and find to which double coset Nt_1 belongs. Nt_1 will be a new double coset, denoted by [1], so thirteen symmetric generators will go to [1].

Nt_1N

In order to find how many single coset [1] contains, we must first find $N^{(1)}$. Then the number of single coset in [1] is equal to $\frac{|N|}{|N^{(1)}|}$. Now,

$$N^{(1)} = N^1\langle e, (2,9,13,6), (3,4,12,11)(5,7,10,8)\rangle$$

on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ so the number of the single cosets in Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{52}{4} = 13$. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:

$$\mathbb{O} = \{1\}, \{2,9,13,6\}, \{3,4,12,11\}, \text{and}\{5,7,10,8\}$$

We take t_1, t_2, t_3, and t_5 from each orbit, respectively, and find to which double coset Nt_1t_1, Nt_1t_2, Nt_1t_3, and Nt_1t_5 belong to. Now $Nt_1t_1 = N \in [*]$, so one element will go back to [*]. On the other hand, three symmetric generators will go to new double cosets Nt_1t_2 denoted by [12], Nt_1t_3 denoted by [13], and Nt_1t_5 denoted by [15].

Nt_1t_2N

Now Nt_1t_2N in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(12)} = \langle Id(N)\rangle$. Then, $|N^{(12)}| = 1$ so the number of single cosets in $N^{(12)}$ is $\frac{|N|}{|N^{(12)}|} = \frac{52}{1} = 52$. The orbits of $N^{(12)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and see which double coset $Nt_1t_2t_i$ belongs to. We have

$$Nt_1t_2t_1 \in [121]$$
$$Nt_1t_2t_2 = Nt_1 \in [1]$$
$$Nt_1t_2t_3 \in [123]$$
$$Nt_1t_2t_4 \in [124]$$

$$
\begin{aligned}
Nt_1t_2t_5 &\in [125] \\
t_1t_2t_6 &= yt_4t_3t_4 \\
\implies Nt_1t_2t_6 &= Nt_4t_3t_4 \in [121] \\
t_1t_2t_7 &= x^2y^{-1}t_1t_6 \\
\implies Nt_1t_2t_7 &= Nt_1t_6 \in [12] \\
Nt_1t_2t_8 &\in [128] \\
Nt_1t_2t_9 &\in [129] \\
t_1t_2t_{10} &= xyx^{-1}t_1t_9 \\
\implies Nt_1t_2t_{10} &= Nt_1t_9 \in [12] \\
t_1t_2t_{11} &\in [1211] \\
Nt_1t_2t_{12} &\in [1212] \\
t_1t_2t_{13} &\in [1213]
\end{aligned}
$$

The new double cosets $Nt_1t_2t_1N$, $Nt_1t_2t_3N$, $Nt_1t_2t_4N$, $Nt_1t_2t_5N$, $Nt_1t_2t_8N$, $Nt_1t_2t_9N$, $Nt_1t_2t_{11}N$, $Nt_1t_2t_{12}N$, and $Nt_1t_2t_{13}N$ are denoted by $[121], [123], [124], [125], [128], [129], [1211], [1212]$, and $[1213]$. And $Nt_1t_2t_1$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_5$, $Nt_1t_2t_8$, $Nt_1t_2t_9$, $Nt_1t_2t_{11}$, $Nt_1t_2t_{12}$, and $Nt_1t_2t_{13}$ is its representative right coset.

$\boldsymbol{Nt_1t_3N}$

Now Nt_1t_3N is a new double coset. However, $N^{(13)} = N^{13} = \langle Id(N) \rangle$. Only identity e will fix 1 and 3. Therefore the number of the single cosets in Nt_1t_3N is $\frac{|N|}{|N^{(13)}|} = \frac{52}{1} = 52$.

The orbits of $N^{(13)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_3t_i$ belongs to.
We have the following:

$$
\begin{aligned}
t_1t_3t_1 &= xyxt_8t_9t_5 \\
\implies Nt_1t_3t_1 &= Nt_8t_9t_5 \in [1211]
\end{aligned}
$$

$$
\begin{aligned}
t_1t_3t_2 &= yx^{-1}yxt_{11}t_3t_6\\
\implies Nt_1t_3t_2 &= Nt_{11}t_3t_6 \in [1213]\\
Nt_1t_3t_3 &= Nt_1 \in [1]\\
t_1t_3t_4 &= yx^3t_4t_{12}t_3\\
\implies Nt_1t_3t_4 &= Nt_4t_{12}t_3 \in [129]\\
t_1t_3t_5 &= yx^2y^{-1}t_9t_7\\
\implies Nt_1t_3t_5 &= Nt_9t_7 \in [13]\\
t_1t_3t_6 &\in [136]\\
t_1t_3t_7 &\in [137]\\
t_1t_3t_8 &= y^{-1}xy^{-1}x^{-1}t_{11}t_{10}t_7\\
\implies Nt_1t_3t_8 &= Nt_{11}t_{10}t_7 \in [125]\\
Nt_1t_3t_9 &\in [139]\\
t_1t_3t_{10} &\in [1310]\\
t_1t_3t_{11} &= y^2x^3t_6t_7t_4\\
\implies Nt_1t_3t_{11} &= Nt_6t_7t_4 \in [1212]\\
t_1t_3t_{12} &= y^{-1}xyt_1t_9t_1\\
\implies Nt_1t_3t_{12} &= Nt_1t_9t_1 \in [121]\\
t_1t_3t_{13} &= x^3t_8t_{11}t_9\\
\implies Nt_1t_3t_{13} &= Nt_8t_{11}t_9 \in [136]
\end{aligned}
$$

The new double cosets $Nt_1t_3t_6N$, $Nt_1t_3t_7N$, $Nt_1t_3t_9N$ and $Nt_1t_3t_{10}N$ are denoted by $[136], [137], [139]$, and $[1310]$ where $Nt_1t_3t_6$, $Nt_1t_3t_7$, $Nt_1t_3t_9$, and $Nt_1t_3t_{10}$ is its representative right coset.

Nt_1t_5N

Also, Nt_1t_5N is a new double coset with $N^{(15)} = \langle Id(N)\rangle$. Only identity will fix 1 and 5. Hence, the number of single cosets in Nt_1t_5N is $\frac{|N|}{|N^{(15)}|} = \frac{52}{1} = 52$. The orbits of $N^{(15)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_i$ belongs to.

We have:

$$\begin{aligned}
t_1t_5t_1 &= xt_9t_{11}t_5\\
\implies Nt_1t_5t_1 &= Nt_9t_{11}t_5 \in [1310]\\
t_1t_5t_2 &= y^{-1}xy^{-1}x^{-1}t_2t_4t_{10}\\
\implies Nt_1t_5t_2 &= Nt_2t_4t_{10} \in [139]\\
t_1t_5t_3 &= yxy^{-1}t_8t_3t_8\\
\implies Nt_1t_5t_3 &= Nt_8t_3t_8 \in [121]\\
t_1t_5t_4 &= yxyt_4t_{12}t_6\\
\implies Nt_1t_5t_4 &= Nt_4t_{12}t_6 \in [1211]\\
Nt_1t_5t_5 &= Nt_1 \in [1]\\
t_1t_5t_6 &= y^{-1}x^{-2}t_{13}t_5t_{13}\\
\implies Nt_1t_5t_6 &= Nt_{13}t_5t_{13} \in [121]\\
t_1t_5t_7 &= x^2y^2t_{11}t_8t_2\\
\implies Nt_1t_5t_7 &= Nt_{11}t_8t_2 \in [137]\\
t_1t_5t_8 &= x^{-3}t_{13}t_{10}t_6\\
\implies Nt_1t_5t_8 &= Nt_{13}t_{10}t_6 \in [1310]\\
t_1t_5t_9 &= xyxt_7t_4t_8\\
\implies Nt_1t_5t_9 &= Nt_7t_4t_8 \in [139]\\
Nt_1t_5t_{10} &\in [1510]\\
Nt_1t_5t_{11} &\in [1511]\\
Nt_1t_5t_{12} &\in [1512]\\
t_1t_5t_{13} &= xyt_8t_3t_6\\
\implies Nt_1t_5t_{13} &= Nt_8t_3t_6 \in [124]
\end{aligned}$$

The new double cosets $Nt_1t_5t_{10}N$, $Nt_1t_5t_{11}N$, and $Nt_1t_5t_{12}N$ are denoted by [1510], [1511], and [1512] where $Nt_1t_5t_{10}$, $Nt_1t_5t_{11}$, and $Nt_1t_5t_{12}$ is its representative right coset.

$\boldsymbol{Nt_1t_2t_1N}$

Consider $Nt_1t_2t_1N$ in N is a new double coset. We determined how many single cosets are on the double coset. $N^{(121)} = N^{121} = \langle e \rangle$. Now lets find the number of single cosets in $Nt_1t_2t_1N$ is $\frac{|N|}{|N^{(121)}|} = \frac{52}{1} = 52$. The orbits of $N(121)$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}$ $,\{11\},\{12\},\{13\}$. Take a representative call it t_i from each orbit and see which double cosets $Nt_1t_2t_1$ belongs to the following:

$$
\begin{aligned}
Nt_1t_2t_1t_1 &= Nt_1t_2 \in [12] \\
t_1t_2t_1t_2 &= xt_5t_{12} \\
\implies Nt_1t_2t_1t_2 &= Nt_5t_{12} \in [15] \\
t_1t_2t_1t_3 &= x^{-1}y^{-1}xt_2t_4t_{11} \\
\implies Nt_1t_2t_1t_3 &= Nt_2t_4 \in [1310] \\
t_1t_2t_1t_4 &= x^{-1}t_1t_{11} \\
\implies Nt_1t_2t_1t_4 &= Nt_1t_{11} \in [13] \\
t_1t_2t_1t_5 &= y^2t_8t_{10}t_3 \\
\implies Nt_1t_2t_1t_5 &= Nt_8t_{10}t_3 \in [139] \\
t_1t_2t_1t_6 &= y^{-1}xy^{-1}t_{12}t_4t_1 \\
\implies Nt_1t_2t_1t_6 &= Nt_{12}t_4t_1 \in [124] \\
t_1t_2t_1t_7 &\in [1217] \\
t_1t_2t_1t_8 &= y^2x^3t_8t_4t_{12} \\
\implies Nt_1t_2t_1t_8 &= Nt_8t_4t_{12} \in [1510] \\
t_1t_2t_1t_9 &= y^2t_1t_2t_1 \\
\implies Nt_1t_2t_1t_9 &= Nt_1t_2t_1 \in [121] \\
t_1t_2t_1t_{10} &= y^2xy{-}1x{-}1t_9t_2 \\
\implies Nt_1t_2t_1t_{10} &= Nt_9t_2 \in [15] \\
t_1t_2t_1t_{11} &= xy^2t_3t_8t_1 \\
\implies Nt_1t_2t_1t_{11} &= Nt_3t_8t_1 \in [1211]
\end{aligned}
$$

$$t_1t_2t_1t_{12} = xy^{-1}xt_4t_3$$
$$\implies Nt_1t_2t_1t_{12} = Nt_4t_3 \in [12]$$
$$t_1t_2t_1t_{13} \in [12113]$$

Nt*₁*t*₂*t*₃*N

Consider $Nt_1t_2t_3N$ in N is a new double coset. However, $N^{(123)} = N^{123} = \langle Id(N)\rangle$. Only identity fix 1,2 and 3. Hence, the number of single cosets in $Nt_1t_2t_3$ is $\frac{|N|}{|N^{(123)}|} = \frac{52}{1} = 52$. The orbits of $N^{(123)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative $t_i$from each orbit and check which double coset $Nt_1t_2t_3t_i$
belongs to.

$$t_1t_2t_3t_1 = y^{-1}x^{-1}t_{12}t_{10}t_7$$
$$\implies Nt_1t_2t_3t_1 = Nt_{12}t_{10}t_7 \in [136]$$
$$t_1t_2t_3t_2 \in [1232]$$
$$Nt_1t_2t_3t_3 = Nt_1t_2 \in [12]$$
$$t_1t_2t_3t_4 = x^{-1}y^{-1}t_{11}t_8t_2$$
$$\implies Nt_1t_2t_3t_4 = Nt_{11}t_8t_2 \in [137]$$
$$t_1t_2t_3t_5 = x^{-2}y^{-1}t_1t_{13}t_7$$
$$\implies Nt_1t_2t_3t_5 = Nt_1t_{13}t_7 \in [128]$$
$$t_1t_2t_3t_6 = x^{-2} * yt_{10}t_{13}t_4$$
$$\implies Nt_1t_2t_3t_6 = Nt_{10}t_{13}t_4 \in [1310]$$
$$t_1t_2t_3t_7 = xyxt_1t_{11}t_2$$
$$\implies Nt_1t_2t_3t_7 = Nt_1t_{11}t_2 \in [139]$$
$$t_1t_2t_3t_8 = xy^{-1}xt_{11}t_2t_7$$
$$\implies Nt_1t_2t_3t_8 = Nt_{11}t_2t_7 \in [1510]$$
$$Nt_1t_2t_3t_9 \in [1239]$$
$$t_1t_2t_3t_{10} \in [12310]$$

$$t_1t_2t_3t_{11} = x^{-1}y^{-1}x^{-1}t_{10}t_{11}t_7$$
$$\implies Nt_1t_2t_3t_{11} = Nt_{10}t_{11}t_7 \in [1211]$$
$$t_1t_2t_3t_{12} = y^2t_7t_{12}t_1$$
$$\implies Nt_1t_2t_3t_{12} = Nt_7t_{12}t_1 \in [125]$$
$$t_1t_2t_3t_{13} = y^2xy^{-1}xt_4t_{12}t_2$$
$$\implies Nt_1t_2t_3t_{13} = Nt_4t_{12}t_2 \in [124]$$

The new double cosets $Nt_1t_2t_3t_2N$, $Nt_1t_2t_3t_9N$, and $Nt_1t_2t_3t_{10}N$ are denoted by [1232], [1239], and [12310] where $Nt_1t_2t_3t_2$, $Nt_1t_2t_3t_9$, and $Nt_1t_2t_3t_{10}$ is its representative right coset.

$Nt_1t_2t_4N$

Consider $Nt_1t_2t_4N$ in N is a new double coset. However, $N^{(124)} = N^{124} = \langle Id(N)\rangle$. Only identity fix 1,2 and 4. Hence, the number of single cosets in $Nt_1t_2t_4$ is $\frac{|N|}{|N^{(124)}|} = \frac{52}{1} = 52$. The orbits of $N^{(124)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative $t_i$from each orbit and check which double coset $Nt_1t_2t_4t_i$ belongs to.

$$t_1t_2t_4t_1 = xy^{-1}xt_{11}t_3t_8t_4$$
$$\implies Nt_1t_2t_4t_1 = Nt_{11}t_3t_8t_4 \in [12310]$$
$$t_1t_2t_4t_2 = x^{-1}y^{-1}x^{-1}t_3t_1t_8$$
$$\implies Nt_1t_2t_4t_2 = Nt_3t_1t_8 \in [139]$$
$$t_1t_2t_4t_3 = yxt_{10}t_{13}t_{11}$$
$$\implies Nt_1t_2t_4t_3 = Nt_{10}t_{13}t_{11} \in [136]$$
$$Nt_1t_2t_4t_4 = Nt_1t_2 \in [12]$$
$$t_1t_2t_4t_5 = yx^{-1}yxt_4t_{12}t_4$$
$$\implies Nt_1t_2t_4t_5 = Nt_4t_{12}t_4 \in [121]$$
$$t_1t_2t_4t_6 = (yxy^{-1})^2t_{11}t_{10}t_8$$
$$\implies Nt_1t_2t_4t_6 = Nt_{10}t_{11}t_8 \in [124]$$

$$t_1t_2t_4t_7 = xy^{-1}x^{-1}t_{12}t_4t_9$$
$$\implies Nt_1t_2t_4t_7 = Nt_{12}t_4t_9 \in [123]$$
$$t_1t_2t_4t_8 = x * yt_7t_6t_{12}$$
$$\implies Nt_1t_2t_4t_8 = Nt_7t_6t_{12} \in [129]$$
$$t_1t_2t_4t_9 = y^2x^2t_{11}t_9t_5$$
$$\implies Nt_1t_2t_4t_9 = Nt_{11}t_9t_5 \in [137]$$
$$t_1t_2t_4t_{10} = x^{-1}t_6t_5t_3$$
$$\implies Nt_1t_2t_4t_{10} = Nt_6t_5t_3 \in [124]$$
$$t_1t_2t_4t_{11} = y^xt_2t_1t_3$$
$$\implies Nt_1t_2t_4t_{11} = Nt_2t_1t_3 \in [1213]$$
$$t_1t_2t_4t_{12} = x^2yt_{12}t_4t_{12}t_3$$
$$\implies Nt_1t_2t_4t_{12} = Nt_{12}t_4t_{12}t_3 \in [1217]$$
$$t_1t_2t_4t_{13} = x^{-1}y^{-1}t_5t_{12}$$
$$\implies Nt_1t_2t_4t_{13} = Nt_5t_{12} \in [15]$$

$Nt_1t_2t_5N$

Consider $Nt_1t_2t_5N$ in N is a new double coset. However, $N^{(125)} = N^{125} = \langle Id(N)\rangle$. Only identity fix 1,2 and 5. The number of single cosets in $Nt_1t_2t_5$ is $\frac{|N|}{|N^{(125)}|} = \frac{52}{1} = 52$. The orbits of $N^{(125)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_5t_i$ belongs to.

$$t_1t_2t_5t_1 = yxt_1t_8t_4$$
$$\implies Nt_1t_2t_5t_1 = Nt_1t_8t_4 \in [1512]$$
$$t_1t_2t_5t_2 = yxyx^{-1}t_5t_{13}t_8$$
$$\implies Nt_1t_2t_5t_2 = Nt_5t_{13}t_8 \in [123]$$
$$t_1t_2t_5t_3 = x^{-1}yx^{-1}t_4t_9t_{12}$$
$$\implies Nt_1t_2t_5t_3 = Nt_4t_9t_{12} \in [1213]$$

$$t_1t_2t_5t_4 = yxyx^{-1}t_{11}t_9$$
$$\implies Nt_1t_2t_5t_4 = Nt_{11}t_9 \in [13]$$
$$Nt_1t_2t_5t_5 = Nt_1t_2 \in [12]$$
$$t_1t_2t_5t_6 = xyt_{10}t_{11_{t_{10}t_3}}$$
$$\implies Nt_1t_2t_5t_6 = Nt_{10}t_{11_{t_{10}t_3}} \in [1217]$$
$$t_1t_2t_5t_7 = yx^{-1}y^{-}1t_8t_9t_{10}t_4$$
$$\implies Nt_1t_2t_5t_7 = Nt_8t_9t_{10}t_4 \in [12310]$$
$$t_1t_2t_5t_8 = x^2t_3t_1t_{11}$$
$$\implies Nt_1t_2t_5t_8 = Nt_3t_1t_{11} \in [136]$$
$$t_1t_2t_5t_9 = yx^2yt_8t_{13}t_6$$
$$\implies Nt_1t_2t_5t_9 = Nt_8t_{13}t_6 \ \in [1211]$$
$$t_1t_2t_5t_{10} = xyxt_9t_{10}t_3$$
$$\implies Nt_1t_2t_5t_{10} = Nt_9t_{10}t_3 \in [128]$$
$$t_1t_2t_5t_{11} = xy^{-1}t_8t_7t_6t_{13}$$
$$\implies Nt_1t_2t_5t_{11} = Nt_8t_7t_6t_{13} \in [1239]$$
$$t_1t_2t_5t_{12} = yxt_{12}t_9t_5$$
$$\implies Nt_1t_2t_5t_{12} = Nt_{12}t_9t_5 \in [1310]$$
$$t_1t_2t_5t_{13} = xyxy^{-1}t_8t_{10}t_1$$
$$\implies Nt_1t_2t_5t_{13} = Nt_8t_{10}t_1 \in [137]$$

$Nt_1t_2t_8N$

Consider $Nt_1t_2t_8N$ in N is a new double coset. However, $N^{(128)} = N^{128} = \langle Id(N)\rangle$. Only identity fix 1,2 and 8. The number of single cosets in $Nt_1t_2t_8$ is $\frac{|N|}{|N^{(128)}|} = \frac{52}{1} = 52$. The orbits of $N^{(128)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_8t_i$ belongs to.

$$t_1t_2t_8t_1 = x^{-1}t_6t_9t_2$$
$$\implies Nt_1t_2t_8t_1 = Nt_6t_9t_2 \in [137]$$
$$t_1t_2t_8t_2 = y^{-1}t_6t_7t_{10}$$
$$\implies Nt_1t_2t_8t_2 = Nt_6t_7t_{10} \in [125]$$
$$t_1t_2t_8t_3 = y^{-1}t_6t_7t_{10}$$
$$\implies Nt_1t_2t_8t_3 = Nt_6t_7t_{10} \in [1217]$$
$$t_1t_2t_8t_4 = yx^2yt_1t_8t_{12}$$
$$\implies Nt_1t_2t_8t_4 = Nt_1t_8t_{12} \in [1511]$$
$$t_1t_2t_8t_5 = x^4t_2t_{12}t_3$$
$$\implies Nt_1t_2t_8t_5 = Nt_2t_{12}t_3 \in [139]$$
$$t_1t_2t_8t_6 = y^2t_6t_7t_8t_2$$
$$\implies Nt_1t_2t_8t_6 = Nt_6t_7t_8t_2 \in [12310]$$
$$t_1t_2t_8t_7 = yx^{-1}y^{-1}x^{-1}t_{11}t_6t_{11}t_3$$
$$\implies Nt_1t_2t_8t_7 = Nt_{11}t_6t_{11}t_3 \in [12113]$$
$$Nt_1t_2t_8t_8 = Nt_1t_2 \in [12]$$
$$t_1t_2t_8t_9 = x^-2y^{-1}t_{11}t_6t_{13}$$
$$\implies Nt_1t_2t_8t_9 = Nt_{11}t_6t_{13} \in [1211]$$
$$t_1t_2t_8t_{10} = x^{-2}yt_1t_{13}t_{12}$$
$$\implies Nt_1t_2t_8t_{10} = Nt_1t_{13}t_{12} \in [123]$$
$$t_1t_2t_8t_{11} = (yx)^2t_7t_2t_{10}t_1$$
$$\implies Nt_1t_2t_8t_{11} = Nt_7t_2t_{10}t_1 \in [12310]$$
$$t_1t_2t_8t_{12} = yx^2yt_8t_{12}t_4$$
$$\implies Nt_1t_2t_8t_{12} = Nt_8t_{12}t_4 \in [1510]$$
$$t_1t_2t_8t_{13} = (x^{-1}, y^{-1})t_6t_8t_1$$
$$\implies Nt_1t_2t_8t_{13} = Nt_6t_8t_1 \in [139]$$

$Nt_1t_2t_9N$

Consider $Nt_1t_2t_9N$ in N is a new double coset. However, $N^{(129)} = N^{129} = \langle Id(N) \rangle$. Only identity fix 1,2 and 9. The number of single cosets in $Nt_1t_2t_9$ is $\frac{|N|}{|N^{(129)}|} = \frac{52}{1} = 52$. The orbits of $N^{(129)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_9t_i$ belongs to.

$$t_1t_2t_9t_1 = y^{-1}xt_{12}t_9$$
$$\implies Nt_1t_2t_9t_1 = Nt_{12}t_9 \in [13]$$
$$t_1t_2t_9t_2 = y^2xt_{11}t_3t_1$$
$$\implies Nt_1t_2t_9t_2 = Nt_{11}t_3t_1 \in [1212]$$
$$t_1t_2t_9t_3 = y^{-1}x^{-1}t_9t_1t_6t_2$$
$$\implies Nt_1t_2t_9t_3 = Nt_9t_1t_6t_2 \in [139]$$
$$t_1t_2t_9t_4 = xyxt_9t_1t_9t_{13}$$
$$\implies Nt_1t_2t_9t_4 = Nt_9t_1t_9t_{13} \in [12310]$$
$$t_1t_2t_9t_5 = xy^2t_6t_{10}t_4$$
$$\implies Nt_1t_2t_9t_5 = Nt_6t_{10}t_4 \in [1512]$$
$$t_1t_2t_9t_6 = x^{-1}t_7t_1t_{13}$$
$$\implies Nt_1t_2t_9t_6 = Nt_7t_1t_{13} \in [1510]$$
$$t_1t_2t_9t_7 = (x^{-1},y^{-1})t_5t_{12}t_{11}$$
$$\implies Nt_1t_2t_9t_7 = Nt_5t_{12}t_{11} \in [1510]$$
$$t_1t_2t_9t_8 = xyxy^{-1}t_3t_{11}t_6t_{11}$$
$$\implies Nt_1t_2t_9t_8 = Nt_3t_{11}t_6t_{11} \in [1232]$$
$$Nt_1t_2t_9t_9 = Nt_1t_2 \in [12]$$
$$t_1t_2t_9t_{10} = xyt_2t_{13}t_7$$
$$\implies Nt_1t_2t_9t_{10} = Nt_2t_{13}t_7 \in [139]$$

$$t_1t_2t_9t_{11} = (yx)^2t_{11}t_{10}t_{13}$$

$$\implies Nt_1t_2t_9t_{11} = Nt_{11}t_{10}t_{13} \in [1212]$$

$$t_1t_2t_9t_{12} = xyxt_1t_4t_8$$

$$\implies Nt_1t_2t_9t_{12} = Nt_1t_4t_8 \in [1310]$$

$$t_1t_2t_9t_{13} = x^{-1}y^{-1}t_7t_6t_4$$

$$\implies Nt_1t_2t_9t_{13} = Nt_7t_6t_4 \in [124]$$

$Nt_1t_2t_{11}N$

Consider $Nt_1t_2t_{11}N$ in N is a new double coset. However, $N^{(1211)} = N^{1211} = \langle Id(N) \rangle$. Only identity fix 1,2 and 11. The number of single cosets in $Nt_1t_2t_{11}$ is $\frac{|N|}{|N^{(1211)}|} = \frac{52}{1} = 52$. The orbits of $N^{(1211)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are: $\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_{11}t_i$ belongs to.

$$t_1t_2t_{11}t_1 = y^2x^2t_{12}t_6$$

$$\implies Nt_1t_2t_{11}t_1 = Nt_{12}t_6 \in [15]$$

$$t_1t_2t_{11}t_2 = xyx^{-1}t_5t_6t_7$$

$$\implies Nt_1t_2t_{11}t_2 = Nt_5t_6t_7 \in [123]$$

$$t_1t_2t_{11}t_3 = x^{-1}yt_8t_7t_8t_2$$

$$\implies Nt_1t_2t_{11}t_3 = Nt_8t_7t_8t_2 \in [1217]$$

$$t_1t_2t_{11}t_4 = y^{-1}x^{-2}t_3t_8t_{12}$$

$$\implies Nt_1t_2t_{11}t_4 = Nt_3t_8t_{12} \in [128]$$

$$t_1t_2t_{11}t_5 = x^{-1}yx^{-1}t_7t_4t_{11}$$

$$\implies Nt_1t_2t_{11}t_5 = Nt_7t_4t_{11} \in [137]$$

$$t_1t_2t_{11}t_6 = xyx^{-1}yt_{12}t_{10}t_4$$

$$\implies Nt_1t_2t_{11}t_6 = Nt_{12}t_{10}t_4 \in [139]$$

$$t_1t_2t_{11}t_7 = x^{-1}y^{-1}xt_7t_9$$

$$\implies Nt_1t_2t_{11}t_7 = Nt_7t_9 \in [13]$$

$$t_1t_2t_{11}t_8 = x^{-3}t_{12}t_9t_5$$
$$\implies Nt_1t_2t_{11}t_8 = Nt_{12}t_9t_5 \in [1310]$$
$$t_1t_2t_{11}t_9 = y^2x^2t_{10}t_5t_3$$
$$\implies Nt_1t_2t_{11}t_9 = Nt_{10}t_5t_3 \in [125]$$
$$t_1t_2t_{11}t_{10} = x^{-1}y^{-1}x^{-1}t_1t_6t_{11}t_6$$
$$\implies Nt_1t_2t_{11}t_{10} = Nt_1t_6t_{11}t_6 \in [1232]$$
$$Nt_1t_2t_{11}t_{11} = Nt_1t_2 \in [12]$$
$$t_1t_2t_{11}t_{12} = xy^{-1}t_6t_5t_4t_{11}$$
$$\implies Nt_1t_2t_{11}t_{12} = Nt_6t_5t_4t_{11} \in [1239]$$
$$t_1t_2t_{11}t_{13} = x^2y^2t_{11}t_6t_{11}$$
$$\implies Nt_1t_2t_{11}t_{13} = Nt_{11}t_6t_{11} \in [121]$$

$\boldsymbol{Nt_1t_2t_{12}N}$

Consider $Nt_1t_2t_{12}N$ in N is a new double coset with $N^{(1212)} = N^{1212} = \langle e\rangle$. But $Nt_1t_2t_{12}N$ is not distinct since $Nt_3t_2t_5 \in [1212]$ and $(1,3)(4,13)(5,12)(6,11)(7,10)$ $(8,9) \in N$.Then $N(t_1t_2t_{12})^{(1,3)(4,13)(5,12)(6,11)(7,10)(8,9)} = Nt_3t_2t_5$ and $t_3t_2t_5$ $= f(x^{-1})t_1t_2t_{12}$.
Thus, $(1,3)(4,13)(5,12)(6,11)(7,10)(8,9) \in N^{(1212)}$. We conclude:

$$N^{(1212)} \geq \langle(1,3)(4,13)(5,12)(6,11)(7,10)(8,9)\rangle.$$

Hence $|N^{(1212)}| = 2$ so the number of single cosets in $N^{(1212)}$ is $\frac{|N|}{|N^{(1212)}|} = \frac{52}{4} = 26$.
The orbits of $N^{(1212)}$ are: $\mathbb{O} = \{2\}, \{1,3\}, \{4,13\}, \{5,12\}, \{6,11\}, \{7,10\}, \{8,9\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_{12}t_i$ belongs to.

$$t_1t_2t_{12}t_2 = yx^{-2}yt_5t_4t_3t_{10}$$
$$\implies Nt_1t_2t_{12}t_2 = Nt_5t_4t_3t_{10} \in [1239]$$
$$t_1t_2t_{12}t_1 = y^2x^2t_{11}t_{10}t_3$$
$$\implies Nt_1t_2t_{12}t_1 = Nt_{11}t_{10}t_3 \in [129]$$

$$t_1t_2t_{12}t_4 = x^{-2}yt_5t_4t_5t_{12}$$
$$\implies Nt_1t_2t_{12}t_4 = Nt_5t_4t_5t_{12} \in [1217]$$
$$Nt_1t_2t_{12}t_{12} = Nt_1t_2 \in [12]$$
$$t_1t_2t_{12}t_6 = yx^{-1}yxt_9t_{11}$$
$$\implies Nt_1t_2t_{12}t_6 = Nt_9t_{11} \in [13]$$
$$t_1t_2t_{12}t_7 = yx^2yt_{12}t_7t_{11}$$
$$\implies Nt_1t_2t_{12}t_7 = Nt_{12}t_7t_{11} \in [129]$$
$$t_1t_2t_{12}t_8 = xyxy^{-1}t_6t_9t_{13}$$
$$\implies Nt_1t_2t_{12}t_8 = Nt_6t_9t_{13} \in [1310]$$

$Nt_1t_2t_{13}N$

Consider $Nt_1t_2t_{13}N$ in N is a new double coset. However, $N^{(1213)} = N^{1213} = \langle Id(N)\rangle$. Only identity fix 1,2 and 13. The number of single cosets in $Nt_1t_2t_{13}$ is $\frac{|N|}{|N^{(1213)}|} = \frac{52}{1} = 52$. The orbits of $N^{(1213)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_{13}t_i$ belongs to.

$$t_1t_2t_{13}t_1 = xyx^{-1}t_4t_5t_4t_{10}$$
$$\implies Nt_1t_2t_{13}t_1 = Nt_4t_5t_4t_{10} \in [1217]$$
$$t_1t_2t_{13}t_2 = x^{-1}y^{-1}x^{-1}t_4t_6t_{13}$$
$$\implies Nt_1t_2t_{13}t_2 = Nt_4t_6t_{13} \in [1310]$$
$$t_1t_2t_{13}t_3 = (yx)^2t_1t_2t_{13}$$
$$\implies Nt_1t_2t_{13}t_3 = Nt_1t_2t_{13} \in [1213]$$
$$t_1t_2t_{13}t_4 = xy^2t_2t_1t_2t_9$$
$$\implies Nt_1t_2t_{13}t_4 = Nt_2t_1t_2t_9 \in [1217]$$
$$t_1t_2t_{13}t_5 = xy^{-1}t_2t_1t_{12}$$
$$\implies Nt_1t_2t_{13}t_5 = Nt_2t_1t_{12} \in [124]$$

$$t_1t_2t_{13}t_6 = x^{-2}y^{-1}t_3t_{11}t_9$$
$$\implies Nt_1t_2t_{13}t_6 = Nt_3t_{11}t_9 \in [125]$$
$$t_1t_2t_{13}t_7 = xyx^{-1}yt_{12}t_2$$
$$\implies Nt_1t_2t_{13}t_7 = Nt_{12}t_2 \in [13]$$
$$t_1t_2t_{13}t_8 = x^2y^{-1}t_9t_2t_3$$
$$\implies Nt_1t_2t_{13}t_8 = Nt_9t_2t_3 \in [1510]$$
$$t_1t_2t_{13}t_9 = y^{-1}xt_4t_{11}t_2$$
$$\implies Nt_1t_2t_{13}t_9 = Nt_4t_{11}t_2 \in [1511]$$
$$t_1t_2t_{13}t_{10} = xyt_5t_6t_7t_1$$
$$\implies Nt_1t_2t_{13}t_{10} = Nt_5t_6t_7t_1 \in [12310]$$
$$t_1t_2t_{13}t_{11} = xt_8t_6t_3$$
$$\implies Nt_1t_2t_{13}t_{11} = Nt_8t_6t_3 \in [136]$$
$$t_1t_2t_{13}t_{12} = xy^{-1}t_5t_7t_{10}$$
$$\implies Nt_1t_2t_{13}t_{12} = Nt_5t_7t_{10} \in [136]$$
$$Nt_1t_2t_{13}t_{13} = Nt_1t_2 \in [12]$$

$\boldsymbol{Nt_1t_3t_6N}$

Consider $Nt_1t_3t_6N$ in N is a new double coset. However, $N^{(136)} = N^{136} = \langle Id(N)\rangle$. Only identity fixes 1,3 and 6. Hence, the number of single cosets in $Nt_1t_3t_6$ is $\frac{|N|}{|N^{(136)}|} = \frac{52}{1} = 52$. The orbits of $N^{(136)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative $t_i$from each orbit and check which double coset $Nt_1t_3t_6t_i$ belongs to.

$$t_1t_3t_6t_1 = xyx^{-1}t_2t_1t_{13}t_6$$
$$\implies Nt_1t_3t_6t_1 = Nt_2t_1t_{13}t_6 \in [12310]$$
$$t_1t_3t_6t_2 = y^2t_1t_{10}t_5$$
$$\implies Nt_1t_3t_6t_2 = Nt_1t_{10}t_5 \in [1510]$$
$$t_1t_3t_6t_3 = (x^{-1},y^{-1})t_6t_5t_6t_7$$

$$\implies Nt_1t_3t_6t_3 = Nt_6t_5t_6t_7 \in [12113]$$

$$Nt_1t_3t_6t_4 = x^3t_4t_{10}t_2t_7t_2$$

$$\implies Nt_1t_3t_6t_4 = Nt_{10}t_2t_7t_2 \in [1232]$$

$$t_1t_3t_6t_5 = y^{-1}x^2t_8t_{13}t_{10}$$

$$\implies Nt_1t_3t_6t_5 = Nt_8t_{13}t_{10} \in [124]$$

$$Nt_1t_3t_6t_6 = Nt_1t_3 \in [13]$$

$$t_1t_3t_6t_7 = (x * y)^2t_5t_{10}t_2t_{11}$$

$$\implies Nt_1t_3t_6t_7 = Nt_5t_{10}t_2t_{11} \in [12310]$$

$$t_1t_3t_6t_8 = yxt_{10}t_{11}t_9$$

$$\implies Nt_1t_3t_6t_8 = Nt_{10}t_{11}t_9 \in [1213]$$

$$t_1t_3t_6t_9 = x^2t_3t_2t_{12}$$

$$\implies Nt_1t_3t_6t_9 = Nt_3t_2t_{12} \in [125]$$

$$t_1t_3t_6t_{10} \in [13610]$$

$$t_1t_3t_6t_{11} = xt_8t_7t_9$$

$$\implies Nt_1t_3t_6t_{11} = Nt_8t_7t_9 \in [1213]$$

$$t_1t_3t_6t_{12} = x^{-1}yt_{12}t_{11}t_{10}$$

$$\implies Nt_1t_3t_6t_{12} = Nt_{12}t_{11}t_{10} \in [123]$$

$$t_1t_3t_6t_{13} = x^{-2}t_5t_2$$

$$\implies Nt_1t_2t_4t_{13} = Nt_5t_2 \in [13]$$

$\boldsymbol{Nt_1t_3t_7N}$

Consider $Nt_1t_3t_7N$ in N is a new double coset. However, $N^{(137)} = N^{137} = \langle Id(N)\rangle$. Only identity fix 1,3 and 7. Hence, the number of single cosets in $Nt_1t_3t_7$ is $\frac{|N|}{|N^{(137)}|} = \frac{52}{1} = 52$. The orbits of $N^{(136)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are: $\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative $t_i$from each orbit and check which double coset $Nt_1t_3t_7t_i$ belongs to.

$$t_1t_3t_7t_1 = y^{-1}t_{10}t_{12}t_5$$

$$\implies Nt_1t_3t_7t_1 = Nt_{10}t_{12}t_5 \in [139]$$

$$t_1t_3t_7t_2 = yxy^{-1}t_2t_7t_{11}$$

$$\implies Nt_1t_3t_7t_2 = Nt_2t_7t_{11} \in [128]$$

$$t_1t_3t_7t_3 = (yx)^2t_{11}t_{10}t_8$$

$$\implies Nt_1t_3t_7t_3 = Nt_{11}t_{10}t_8 \in [124]$$

$$Nt_1t_3t_7t_4 = (x^{-1}, y^{-1})t_7t_{13}t_1$$

$$\implies Nt_1t_3t_7t_4 = Nt_7t_{13}t_1 \in [1510]$$

$$t_1t_3t_7t_5 = y^2x^2t_{10}t_{11}t_{10}t_3$$

$$\implies Nt_1t_3t_7t_5 = Nt_{10}t_{11}t_{10}t_3 \in [1217]$$

$$t_1t_3t_7t_6 = yx^{-1}y^{-1}x^{-1}t_7t_8t_{11}$$

$$\implies Nt_1t_3t_7t_6 = Nt_7t_8t_{11} \in [125]$$

$$Nt_1t_3t_7t_7 = Nt_1t_3 \in [13]$$

$$t_1t_3t_7t_8 = yx^{-1}yt_{12}t_5$$

$$\implies Nt_1t_3t_7t_8 = Nt_{12}t_5 \in [15]$$

$$t_1t_3t_7t_9 = x^{-2}yt_{13}t_{12}t_{11}t_5$$

$$\implies Nt_1t_3t_7t_9 = Nt_{13}t_{12}t_{11}t_5 \in [1239]$$

$$t_1t_3t_7t_{10} = y^xt_{12}t_7t_2$$

$$\implies Nt_1t_3t_7t_{10} = Nt_{12}t_7t_2 \in [123]$$

$$t_1t_3t_7t_{11} = x^{-1}y^{-1}t_5t_{13}t_7$$

$$\implies Nt_1t_3t_7t_{11} = Nt_5t_{13}t_7 \in [1211]$$

$$t_1t_3t_7t_{12} = t_{10}t_5t_{13}t_4$$

$$\implies Nt_1t_3t_7t_{12} = Nt_{10}t_5t_{13}t_4 \in [12310]$$

$$t_1t_3t_7t_{13} = y^{-1}x^{-2}t_6t_2t_8$$

$$\implies Nt_1t_2t_7t_{13} = Nt_6t_2t_8 \in [1512]$$

$\boldsymbol{Nt_1t_3t_9N}$

From above $Nt_1t_3t_9N$ in N is a new double coset with $N^{(139)} = N^{139} = \langle e \rangle$. Now we are going to check how many new single cosets are in the double cosets in

$Nt_1t_3t_9$ is $\frac{|N|}{|N^{(139)}|} = \frac{52}{1} = 52$. The orbits of $N^{(139)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_3t_9t_i$ belongs to.

$$t_1t_3t_9t_1 = y^2x^2t_{13}t_4$$
$$\implies Nt_1t_3t_9t_1 = Nt_{13}t_4 \in [15]$$
$$t_1t_3t_9t_2 = yx^2t_3t_2t_{13}$$
$$\implies Nt_1t_3t_9t_2 = Nt_3t_2t_{13} \in [124]$$
$$t_1t_3t_9t_3 = xyx^{-1}yt_7t_2t_{10}t_1$$
$$\implies Nt_1t_3t_9t_3 = Nt_7t_2t_{10}t_1 \in [12310]$$
$$t_1t_3t_9t_4 = y^{-1}x^3t_5t_{11}$$
$$\implies Nt_1t_3t_9t_4 = Nt_5t_{11} \in [15]$$
$$t_1t_3t_9t_5 = yx^{-2}t_5t_7t_{11}$$
$$\implies Nt_1t_3t_9t_5 = Nt_5t_7t_{11} \in [137]$$
$$t_1t_3t_9t_6 = y^{-1}x^2t_2t_1t_7$$
$$\implies Nt_1t_3t_9t_6 = Nt_2t_1t_7 \in [129]$$
$$t_1t_3t_9t_7 = (y * x)^2t_{12}t_{11}t_2$$
$$\implies Nt_1t_3t_9t_7 = Nt_{12}t_{11}t_2 \in [1211]$$
$$t_1t_3t_9t_8 = (y^{-1}, x^{-1})t_9t_{10}t_3$$
$$\implies Nt_1t_3t_9t_8 = Nt_9t_{10}t_3 \in [128]$$
$$Nt_1t_3t_9t_9 = Nt_1t_3 \in [13]$$
$$t_1t_3t_9t_{10} = xy^{-1}xt_1t_9t_4$$
$$\implies Nt_1t_3t_9t_{10} = Nt_1t_9t_5 \in [123]$$
$$t_1t_3t_9t_{11} = y^2x^{-1}t_7t_8t_7$$
$$\implies Nt_1t_3t_9t_{11} = Nt_7t_8t_7 \in [121]$$
$$t_1t_3t_9t_{12} = (y, x^{-1})t_6t_1t_{10}$$

$$\implies Nt_1t_3t_9t_{12} = Nt_6t_1t_{10} \in [128]$$

$$t_1t_3t_9t_{13} = yxt_{13}t_6t_2$$

$$\implies Nt_1t_3t_9t_{13} = Nt_{13}t_6t_2 \in [1511]$$

$Nt_1t_3t_{10}N$

From above $Nt_1t_3t_{10}N$ in N is a new double coset with $N^{(1310)} = N^{1310} = \langle e \rangle$. Now we are going to check how many new single cosets are in the double cosets in $Nt_1t_3t_{10}$ is $\frac{|N|}{|N^{(1310)}|} = \frac{52}{1} = 52$. The orbits of $N^{(1310)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_3t_{10}t_i$ belongs to.

$$t_1t_3t_{10}t_1 = y^{-1}x^2t_4t_{12}t_{10}$$

$$\implies Nt_1t_3t_{10}t_1 = Nt_4t_{12}t_{10} \in [125]$$

$$t_1t_3t_{10}t_2 = yt_{13}t_1t_{13}$$

$$\implies Nt_1t_3t_{10}t_2 = Nt_{13}t_1t_{13} \in [121]$$

$$t_1t_3t_{10}t_3 = (x^{-1}, y^{-1})t_2t_{12}t_1t_8$$

$$\implies Nt_1t_3t_{10}t_3 = Nt_2t_{12}t_1t_8 \in [13610]$$

$$t_1t_3t_{10}t_4 = y^{-1} * x^{-1}t_1t_6t_2$$

$$\implies Nt_1t_3t_{10}t_4 = Nt_1t_6t_2 \in [129]$$

$$t_1t_3t_{10}t_5 = t_9t_7t_{13}$$

$$\implies Nt_1t_3t_{10}t_5 = Nt_9t_7t_{13} \in [1310]$$

$$t_1t_3t_{10}t_6 = x^{-1}t_6t_{10}$$

$$\implies Nt_1t_3t_{10}t_6 = Nt_6t_{10} \in [15]$$

$$t_1t_3t_{10}t_7 = y^{-1}x^{-2}t_8t_{13}t_5$$

$$\implies Nt_1t_3t_{10}t_7 = Nt_8t_{13}t_5 \in [123]$$

$$t_1t_3t_{10}t_8 = x^{-2}t_4t_{12}t_6$$

$$\implies Nt_1t_3t_{10}t_8 = Nt_4t_{12}t_6 \in [1211]$$

$$t_1t_3t_{10}t_9 = t_4t_9t_5$$
$$\implies Nt_1t_3t_{10}t_9 = Nt_4t_9t_5 \in [12310]$$
$$Nt_1t_3t_{10}t_{10} = Nt_1t_3 \in [13]$$
$$t_1t_3t_{10}t_{11} = (x^{-1}, y^{-1})t_2t_7t_5$$
$$\implies Nt_1t_3t_{10}t_{11} = Nt_2t_7t_5 \in [1212]$$
$$t_1t_3t_{10}t_{12} = x^{-1}yx^{-1}t_{11}t_{12}t_{10}$$
$$\implies Nt_1t_3t_{10}t_{12} = Nt_{11}t_{12}t_{10} \in [1213]$$
$$t_1t_3t_{10}t_{13} = x^{-2}t_9t_2$$
$$\implies Nt_1t_3t_{10}t_{13} = Nt_9t_2 \in [15]$$

$\boldsymbol{Nt_1t_5t_{10}N}$

From above $Nt_1t_5t_{10}N$ in N is a new double coset with $N^{(1510)} = N^{1510} = \langle e \rangle$. Now we are going to check how many new single cosets are in the double cosets in $Nt_1t_5t_{10}$ is $\frac{|N|}{|N^{(1510)}|} = \frac{52}{1} = 52$. The orbits of $N^{(1510)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_5t_{10}t_i$ belongs to.

$$t_1t_5t_{10}t_1 = y^2x^2t_8t_7t_8$$
$$\implies Nt_1t_5t_{10}t_1 = Nt_8t_7t_8 \in [121]$$
$$t_1t_5t_{10}t_2 = y^2t_3t_2t_1t_8$$
$$\implies Nt_1t_5t_{10}t_2 = Nt_3t_2t_1t_8 \in [1239]$$
$$t_1t_5t_{10}t_3 = x^{-2}y^{-1}t_1t_{13}t_1t_2$$
$$\implies Nt_1t_5t_{10}t_3 = Nt_1t_{13}t_1t_2 \in [12113]$$
$$t_1t_5t_{10}t_4 = xyxy^{-1}t_8t_3t_7$$
$$\implies Nt_1t_5t_{10}t_4 = Nt_8t_3t_7 \in [129]$$
$$t_1t_5t_{10}t_5 = y^2x^2t_7t_8t_1$$
$$\implies Nt_1t_5t_{10}t_5 = Nt_7t_8t_1 \in [128]$$

$$t_1t_5t_{10}t_6 = yx^{-1}y^{-1}t_5t_{13}t_4$$

$$\implies Nt_1t_5t_{10}t_6 = Nt_5t_{13}t_4 \in [129]$$

$$t_1t_5t_{10}t_7 = yx^2t_9t_5t_{12}$$

$$\implies Nt_1t_5t_{10}t_7 = Nt_9t_5t_{12} \in [1511]$$

$$t_1t_5t_{10}t_8 = x^2y^2t_{12}t_7t_2t_{11}$$

$$\implies Nt_1t_5t_{10}t_8 = Nt_{12}t_7t_2t_{11} \in [1239]$$

$$t_1t_5t_{10}t_9 = yx^{-1}t_{13}t_5t_8$$

$$\implies Nt_1t_5t_{10}t_9 = Nt_{13}t_5t_8 \in [1213]$$

$$Nt_1t_5t_{10}t_{10} = Nt_1t_5 \in [15]$$

$$t_1t_5t_{10}t_{11} = xyxt_4t_5t_6$$

$$\implies Nt_1t_5t_{10}t_{11} = Nt_4t_5t_6 \in [123]$$

$$t_1t_5t_{10}t_{12} = yx^{-1}y^{-1}x^{-1}t_{10}t_7t_1$$

$$\implies Nt_1t_5t_{10}t_{12} = Nt_{10}t_7t_1 \in [137]$$

$$t_1t_5t_{10}t_{13} = xyx^{-1}yt_1t_{12}t_9$$

$$\implies Nt_1t_5t_{10}t_{13} = Nt_1t_{12}t_9 \in [136]$$

$\boldsymbol{Nt_1t_5t_{11}N}$

Consider $Nt_1t_5t_{11}N$ in N is a new double coset with $N^{(1511)} = N^{1511} = \langle e \rangle$. But $Nt_1t_5t_{11}N$ is not distinct since $Nt_6t_2t_9 \in [1511]$ and $(1,6)(2,5)(3,4)(7,13)(8,12)$ $(9,11) \in N$.Then $N(t_1t_5t_{11})^{(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)} = Nt_6t_2t_9$. Thus, $N^{(1511)} \geq \langle (1,6)(2,5)(3,4)(7,13)(8,12)(9,11) \rangle$. Hence $|N^{(1511)}| = 2$ so the number of single cosets in $N^{(1511)}$ is $\frac{|N|}{|N^{(1511)}|} = \frac{52}{2} = 26$.
The orbits of $N^{(1511)}$ are: $\mathbb{O} = \{10\}, \{1,6\}\{2,5\}, \{3,4\}\{7,13\}, \{8,12\}\{9,11\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_{11}t_i$ belongs to.

$$t_1t_5t_{11}t_{10} = y^{-1}xy^{-1}x^{-1}t_4t_{11}t_2$$

$$\implies Nt_1t_5t_{11}t_{10} = Nt_4t_{11}t_2 \in [1511]$$

$$t_1t_5t_{11}t_1 = x^2y^{-1}t_6t_3t_7$$
$$\implies Nt_1t_5t_{11}t_1 = Nt_6t_3t_7 \in [139]$$
$$t_1t_5t_{11}t_2 = x^{-2}yt_3t_{11}t_8$$
$$\implies Nt_1t_5t_{11}t_2 = Nt_3t_{11}t_8 \in [1213]$$
$$t_1t_5t_{11}t_3 = y^2xyxt_9t_5t_{13}$$
$$\implies Nt_1t_5t_{11}t_3 = Nt_9t_5t_{13} \in [1510]$$
$$t_1t_5t_{11}t_7 = y^{-1}x^2t_{11}t_5t_1$$
$$\implies Nt_1t_5t_{11}t_7 = Nt_{11}t_5t_1 \in [1512]$$
$$t_1t_5t_{11}t_8 = y^2x^3t_6t_{11}t_2$$
$$\implies Nt_1t_5t_{11}t_8 = Nt_6t_{11}t_2 \in [128]$$
$$Nt_1t_5t_{11}t_{11} = Nt_1t_5 \in [15]$$

$Nt_1t_5t_{12}N$

Now $Nt_1t_5t_{12}N$ in N is a new double coset with $N^{(1512)} = N^{1512} = \langle Id(N)\rangle$. But $Nt_1t_5t_{12}N$ is not distinct since $Nt_3t_{12}t_5 \in [1512]$ and $(1,3)(4,13)(5,12)(6,11)(7,10)$ $(8,9) \in N$.Then $N(t_1t_5t_{12})^{(1,3)(4,13)(5,12)(6,11)(7,10)(8,9)} = Nt_3t_{12}t_5$.
Thus, $(1,3)(4,13)(5,12)(6,11)(7,10)(8,9) \in N^{(1512)}$. We conclude:

$$N^{(1512)} \geq \langle (1,3)(4,13)(5,12)(6,11)(7,10)(8,9)\rangle.$$

Hence $|N^{(1512)}| = 2$ so the number of single cosets in $N^{(1512)}$ is $\frac{|N|}{|N^{(1512)}|} = \frac{52}{2} = 26$.
The orbits of $N^{(1512)}$ are: $\mathbb{O} = \{2\}, \{1,3\}, \{4,13\}, \{5,12\}, \{6,11\}, \{7,10\}, \{8,9\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_{12}t_i$ belongs to.

$$t_1t_5t_{12}t_2 = xyxy^{-1}t_4t_3t_2t_3$$
$$\implies Nt_1t_5t_{12}t_2 = Nt_4t_3t_2t_3 \in [1232]$$
$$t_1t_5t_{12}t_1 = y^{-1}xt_1t_9t_7$$
$$\implies Nt_1t_5t_{12}t_1 = Nt_1t_9t_7 \in [125]$$

$$t_1t_5t_{12}t_4 = y^2xt_8t_7t_{13}$$
$$\implies Nt_1t_5t_{12}t_4 = Nt_8t_7t_{13} \in [129]$$
$$Nt_1t_5t_{12}t_{12} = Nt_1t_5 \in [15]$$
$$t_1t_5t_{12}t_6 = y^2xt_1t_5t_{12}$$
$$\implies Nt_1t_5t_{12}t_6 = Nt_1t_5t_{12} \in [1512]$$
$$t_1t_5t_{12}t_7 = yt_6t_4t_{13}$$
$$\implies Nt_1t_5t_{12}t_7 = Nt_6t_4t_{13} \in [137]$$
$$t_1t_5t_{12}t_8 = yt_{12}t_5t_1$$
$$\implies Nt_1t_5t_{12}t_8 = Nt_{12}t_5t_1 \in [1511]$$

$Nt_1t_2t_1t_7N$

From above $Nt_1t_2t_1t_7N$ in N is a new double coset with $N^{(1217)} = N^{1217} = \langle e \rangle$. Now we are going to check how many new single cosets are in the double cosets in $Nt_1t_2t_1t_7$ is $\frac{|N|}{|N^{(1217)}|} = \frac{52}{1} = 52$. The orbits of $N^{(1217)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_1t_7t_i$ belongs to.

$$t_1t_2t_1t_7t_1 = y^{-1}t_4t_{12}t_2$$
$$\implies Nt_1t_2t_1t_7t_1 = Nt_4t_{12}t_2 \in [124]$$
$$t_1t_2t_1t_7t_2 = y^{-1}x^{-1}t_3t_4t_1$$
$$\implies Nt_1t_2t_1t_7t_2 = Nt_3t_4t_1 \in [1212]$$
$$Nt_1t_2t_1t_7t_3 = xy^{-1}t_9t_1t_9t_{13}$$
$$\implies Nt_1t_2t_1t_7t_3 = Nt_9t_1t_9t_{13} \in [1217]$$
$$t_1t_2t_1t_7t_4 = y^xt_7t_9t_{12}t_3$$
$$\implies Nt_1t_2t_1t_7t_4 = Nt_7t_9t_{12}t_3 \in [13610]$$

$$t_1t_2t_1t_7t_5 = xyt_2t_{10}t_2t_{11}$$
$$\implies Nt_1t_2t_1t_7t_5 = Nt_2t_{10}t_2t_{11} \in [1217]$$
$$t_1t_2t_1t_7t_6 = y^{-1}x^2t_8t_7t_{11}$$
$$\implies Nt_1t_2t_1t_7t_6 = Nt_8t_7t_{11} \in [1211]$$
$$Nt_1t_2t_1t_7t_7 = Nt_1t_2t_1 \in [121]$$
$$t_1t_2t_1t_7t_8 = xy^{-1}xt_5t_{13}t_9$$
$$\implies Nt_1t_2t_1t_7t_8 = Nt_5t_{13}t_9 \in [128]$$
$$t_1t_2t_1t_7t_9 = yx^{-2}yt_5t_7t_{11}$$
$$\implies Nt_1t_2t_1t_7t_9 = Nt_5t_7t_{11} \in [137]$$
$$t_1t_2t_1t_7t_{10} = x^{-1}y^{-1}xt_5t_6t_9$$
$$\implies Nt_1t_2t_1t_7t_{10} = Nt_5t_6t_9 \in [125]$$
$$t_1t_2t_1t_7t_{11} = x^{-2}y^{-1}t_{11}t_{12}t_{10}$$
$$\implies Nt_1t_2t_1t_7t_{11} = Nt_{11}t_{12}t_{10} \in [1213]$$
$$t_1t_2t_1t_7t_{12} = (xy)^2t_2t_1t_3$$
$$\implies Nt_1t_2t_1t_7t_{12} = Nt_2t_1t_3 \in [1213]$$
$$t_1t_2t_1t_7t_{13} = y^{-1}x^2t_2t_{10}t_1$$
$$\implies Nt_1t_2t_1t_7t_{13} = Nt_2t_{10}t_1 \in [129]$$

$Nt_1t_2t_1t_{13}N$

Also $Nt_1t_2t_1t_{13}N$ in N is a new double coset. With $N^{(12113)} = N^{12113} = \langle Id(N)\rangle$. But $Nt_1t_2t_1t_{13}N$ is not distinct. Now $Nt_9t_8t_9t_{10} \in [12113]$ since $(1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \in N$ and $N(t_1t_2t_1t_{13})^{(1,9)(2,8)(3,7)(4,6)(10,13)(11,12)} = Nt_9t_8t_9t_{10}$. Thus, $(1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \in N^{(12113)}$. We conclude:

$$N^{(12113)} \geq \langle (1,9)(2,8)(3,7)(4,6)(10,13)(11,12)\rangle.$$

Hence $|N^{(12113)}| = 2$ so the number of single cosets in $N^{(12113)}$ is $\frac{|N|}{|N^{(12113)}|} = \frac{52}{2} = 26$. The orbits of $N^{(12113)}$ are:
$\mathbb{O} = \{5\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{10,13\}, \{11,12\}$.

Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_1t_{13}t_i$ belongs to. We have:

$$t_1t_2t_1t_{13}t_5 = yx^{-1}yt_{11}t_3t_{11}t_6$$
$$\implies Nt_1t_2t_1t_{13}t_5 = Nt_{11}t_3t_{11}t_6 \in [12113]$$
$$t_1t_2t_1t_{13}t_1 = x^{-2}y^{-1}t_6t_7t_8t_2$$
$$\implies Nt_1t_2t_1t_{13}t_1 = Nt_6t_7t_8t_2 \in [12310]$$
$$Nt_1t_2t_1t_{13}t_2 = xy^{-1}x^{-1}t_{13}t_{12}t_{11}t_{12}$$
$$\implies Nt_1t_2t_1t_{13}t_2 = Nt_{13}t_{12}t_{11}t_{12} \in [1232]$$
$$t_1t_2t_1t_{13}t_3 = x^2t_7t_2t_{11}$$
$$\implies Nt_1t_2t_1t_{13}t_3 = Nt_7t_2t_{11} \in [128]$$
$$t_1t_2t_1t_{13}t_4 = (x^{-1}, y^{-1})t_6t_4t_1$$
$$\implies Nt_1t_2t_1t_{13}t_4 = Nt_6t_4t_1 \in [136]$$
$$Nt_1t_2t_1t_{13}t_{13} = Nt_1t_2t_1 \in [121]$$
$$t_1t_2t_1t_{13}t_{11} = xyx^{-1}t_9t_{13}t_5$$
$$\implies Nt_1t_2t_1t_{13}t_{11} = Nt_9t_{13}t_5 \in [1510]$$

$\boldsymbol{Nt_1t_2t_3t_2N}$

Also $Nt_1t_2t_3t_2N$ in N is a new double coset. With $N^{(1232)} = N^{1232} = \langle Id(N)\rangle$. But $Nt_1t_2t_3t_2N$ is not distinct. Now $Nt_5t_4t_3t_4 \in [1232]$ since $(1,5)(2,4)(6,13)(7,12)(8,11)(9,10) \in N$ and $N(t_1t_2t_3t_2)^{(1,5)(2,4)(6,13)(7,12)(8,11)(9,10)}$ $= Nt_5t_4t_3t_4$. Thus, $(1,5)(2,4)(6,13)(7,12)(8,11)(9,10) \in N^{(1232)}$. We conclude:

$$N^{(1232)} \geq \langle (1,5)(2,4)(6,13)(7,12)(8,11)(9,10)\rangle.$$

Hence $|N^{(1232)}| = 2$ so the number of single cosets in $N^{(1232)}$ is $\frac{|N|}{|N^{(1232)}|} = \frac{52}{2} = 26$. The orbits of $N^{(1232)}$ are:
$\mathbb{O} = \{3\}, \{1,5\}, \{2,4\}, \{6,13\}, \{7,12\}, \{8,11\}, \{9,10\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_3t_2t_i$

belongs to. We have:

$$\begin{aligned}
& t_1t_2t_3t_2t_3 = y^2x^2t_5t_4t_7 \\
\implies & Nt_1t_2t_3t_2t_3 = Nt_5t_4t_7 \in [1212] \\
& t_1t_2t_3t_2t_1 = y^2xt_{10}t_3t_4 \\
\implies & Nt_1t_2t_3t_2t_1 = Nt_{10}t_3t_4 \in [1510] \\
& Nt_1t_2t_3t_2t_2 = Nt_1t_2t_3 \in [123] \\
& t_1t_2t_3t_2t_6 = yx^{-1}t_{10}t_2t_4 \\
\implies & Nt_1t_2t_3t_2t_6 = Nt_{10}t_2t_4 \in [125] \\
& t_1t_2t_3t_2t_7 = yx^{-1}t_{10}t_2t_4 \\
\implies & Nt_1t_2t_3t_2t_7 = Nt_{10}t_2t_4 \in [125] \\
& t_1t_2t_3t_2t_8 = t_1t_2t_1 \in [121] \\
\implies & Nt_1t_2t_3t_2t_8 = Nt_{10}t_2t_4 \in [125] \\
& t_1t_2t_3t_2t_9 = yx^{-1}t_{10}t_2t_4 \\
\implies & Nt_1t_2t_3t_2t_9 = Nt_{10}t_2t_4 \in [125]
\end{aligned}$$

$\boldsymbol{Nt_1t_2t_3t_9N}$

Also $Nt_1t_2t_3t_9N$ in N is a new double coset. With $N^{(1239)} = N^{1239} = \langle Id(N)\rangle$. But $Nt_1t_2t_3t_9N$ is not distinct. Now $Nt_7t_6t_5t_{12} \in [1239]$
since $(1,7)(2,6)(3,5)(8,13)(9,12)(10,11) \in N$ and $N(t_1t_2t_3t_9)^{(1,7)(2,6)(3,5)(8,13)(9,12)(10,11)}$
$= Nt_7t_6t_5t_{12}$. Thus, $(1,7)(2,6)(3,5)(8,13)(9,12)(10,11) \in N^{(1239)}$. We conclude:

$$N^{(1239)} \geq \langle (1,7)(2,6)(3,5)(8,13)(9,12)(10,11)\rangle.$$

Hence $|N^{(1239)}| = 2$ so the number of single cosets in $N^{(1239)}$ is $\frac{|N|}{|N^{(1239)}|} = \frac{52}{2} = 26$. The orbits of $N^{(1239)}$ are:
$\mathbb{O} = \{4\}, \{1,7\}, \{2,6\}, \{3,5\}, \{8,13\}, \{9,12\}, \{10,11\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_3t_2t_i$

belongs to. We have:

$$t_1t_2t_3t_9t_4 = y^2x^2t_5t_4t_7$$
$$\implies Nt_1t_2t_9t_2t_4 = Nt_5t_4t_7 \in [1212]$$
$$t_1t_2t_3t_9t_1 = y^2xt_{10}t_3t_4$$
$$\implies Nt_1t_2t_3t_9t_1 = Nt_{10}t_3t_4 \in [1510]$$
$$Nt_1t_2t_3t_9t_2 = yxyt_3t_{12}t_7$$
$$\implies Nt_1t_2t_3t_9t_2 = Nt_3t_{12}t_7 \in [1510]$$
$$t_1t_2t_3t_9t_3 = y^{-1}t_8t_{10}t_1$$
$$\implies Nt_1t_2t_3t_9t_3 = Nt_8t_{10}t_1 \in [137]$$
$$t_1t_2t_3t_9t_8 = x^{-1} * yx^{-1}t_6t_5t_9$$
$$\implies Nt_1t_2t_3t_9t_8 = Nt_6t_5t_9 \in [1211]$$
$$Nt_1t_2t_3t_9t_9 = Nt_1t_2t_3 \in [123]$$
$$t_1t_2t_3t_9t_{10} = y^xt_{13}t_1t_4$$
$$\implies Nt_1t_2t_3t_9t_{10} = Nt_{13}t_1t_4 \in [125]$$

$\boldsymbol{Nt_1t_2t_3t_{10}N}$

From above $Nt_1t_2t_3t_{10}N$ in N is a new double coset with $N^{(12310)} = N^{12310} = \langle Id(N)\rangle$. Now we are going to check how many new single cosets are in the double cosets in $Nt_1t_2t_3t_{10}$ is $\frac{|N|}{|N^{(12310)}|} = \frac{52}{1} = 52$. The orbits of $N^{(12310)}$ on
$\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_3t_{10}t_i$ belongs to.

$$t_1t_2t_3t_{10}t_1 = yx^2yt_9t_{10}t_3$$
$$\implies Nt_1t_2t_{31}t_{10}t_1 = Nt_9t_{10}t_3 \in [128]$$
$$t_1t_2t_3t_{10}t_2 = x^2y^{-1}t_2t_{13}t_{10}$$
$$\implies Nt_1t_2t_3t_{10}t_2 = Nt_2t_{13}t_{10} \in [136]$$

$$t_1t_2t_3t_{10}t_3 = t_4t_7t_{11}$$
$$\implies Nt_1t_2t_3t_{10}t_3 = Nt_4t_7t_{11} \in [1310]$$
$$t_1t_2t_3t_{10}t_4 = xyx^{-1}yt_8t_{11}t_9$$
$$\implies Nt_1t_2t_3t_{10}t_4 = Nt_8t_{11}t_9 \in [136]$$
$$t_1t_2t_3t_{10}t_5 = yx^2t_2t_{10}t_1$$
$$\implies Nt_1t_2t_3t_{10}t_5 = Nt_2t_{10}t_1 \in [129]$$
$$t_1t_2t_3t_{10}t_6 = y^{-1}x^2t_{10}t_{11}t_9$$
$$\implies Nt_1t_2t_3t_{10}t_6 = Nt_{10}t_{11}t_9 \in [1213]$$
$$t_1t_2t_3t_{10}t_7 = xyx^{-1}yt_{10}t_7t_{11}$$
$$\implies Nt_1t_2t_3t_10t_7 = Nt_{10}t_7t_{11} \in [139]$$
$$t_1t_2t_3t_{10}t_8 = xyx^{-1}yt_{10}t_7t_1$$
$$\implies Nt_1t_2t_3t_{10}t_8 = Nt_{10}t_7t_1 \in [137]$$
$$t_1t_2t_3t_{10}t_9 = x^{-1}yx^{-1}t_4t_3t_4t_5$$
$$\implies Nt_1t_2t_3t_{10}t_9 = Nt_4t_3t_4t_5 \in [12113]$$
$$Nt_1t_2t_3t_{10}t_{10} = Nt_1t_2t_3 \in [123]$$
$$t_1t_2t_3t_{10}t_{11} = t_8t_5t_{12}$$
$$\implies Nt_1t_2t_3t_{10}t_{11} = Nt_8t_5t_{12} \in [137]$$
$$t_1t_2t_3t_{10}t_{12} = xy^{-1}x^{-1}t_{12}t_7t_{10}$$
$$\implies Nt_1t_2t_3t_{10}t_{12} = Nt_{12}t_7t_{10} \in [124]$$
$$t_1t_2t_3t_{10}t_{13} = yxy^{-1}t_7t_8t_{11}$$
$$\implies Nt_1t_2t_3t_{10}t_{13} = Nt_7t_8t_{11} \in [125]$$

$\boldsymbol{Nt_1t_3t_6t_{10}N}$

Also $Nt_1t_3t_6t_{10}N$ in N is a new double coset. With $N^{(13610)} = N^{13610}$ $= \langle Id(N)\rangle$. But $Nt_1t_3t_6t_{10}N$ is not distinct. Now $Nt_7t_4t_6t_{13} \in [13610]$ since $(1,7,11,5)(2,12,10,13)(3,4,9,8) \in N$ and $N(t_1t_3t_6t_{10})^{(1,7,11,5)(2,12,10,13)(3,4,9,8)}$ $= Nt_7t_4t_6t_{13}$ so $t_7t_4t_6t_{13} = x^6t_1t_3t_6t_{10}$. Thus, $(1,7,11,5)(2,12,10,13)(3,4,9,8) \in$

$N^{(13610)}$. We conclude:

$$N^{(13610)} \geq \langle (1,7,11,5)(2,12,10,13)(3,4,9,8) \rangle.$$

Hence $|N^{(13610)}| = 4$ so the number of single cosets in $N^{(13610)}$ is $\frac{|N|}{|N^{(13610)}|} = \frac{52}{4} = 13$.
The orbits of $N^{(13610)}$ are:
$\mathbb{O} = \{6\}, \{1,7,11,5\}, \{2,12,10,13\}, \{3,4,9,8\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_3t_6t_{10}t_i$ belongs to. We have:

$$\begin{aligned}
t_1t_3t_6t_{10}t_6 &\in [136106] \\
t_1t_3t_6t_{10}t_1 &= y^{-1}x^{-1}t_4t_3t_4t_{11} \\
\implies Nt_1t_3t_6t_{10}t_1 &= Nt_4t_3t_4t11 \in [1217] \\
Nt_1t_3t_6t_{10}t_{10} &= Nt_1t_3t_6 \in [136] \\
t_1t_3t_6t_{10}t_3 &= x^{-3}t_6t_3t_{12} \\
\implies Nt_1t_3t_6t_{10}t_3 &= Nt_6t_3t_{12} \in [1310]
\end{aligned}$$

$Nt_1t_3t_6t_{10}t_6N$

Also $Nt_1t_3t_6t_{10}t_6N$ in N is a new double coset. With $N^{(136106)} = N^{136106} = \langle Id(N) \rangle$. But $Nt_1t_3t_6t_{10}t_6N$ is not distinct since $Nt_2t_4t_7t_{11}t_7 \in [136106]$ thus $(1,2,3,4,5,6,7,8,9,10,11,12,13) \in N$ and $N(t_1t_3t_6t_{10}t_6)^{(1,2,3,4,5,6,7,8,9,10,11,12,13)} = Nt_2t_4t_7t_{11}t_7$ so $t_2t_4t_7t_{11}t_7 = x^{11}t_1t_3t_6t_{10}t_6$. Therefore, $(1,2,3,4,5,6,7,8,9,10,11,12,13) \in N^{(136106)}$. We conclude:

$$N^{(136106)} \geq \langle (1,2,3,4,5,6,7,8,9,10,11,12,13) \rangle.$$

Hence $|N^{(136106)}| = 13$ so the number of single cosets in $N^{(136106)}$ is $\frac{|N|}{|N^{(136106)}|} = \frac{52}{13} = 4$.
The orbits of $N^{(136106)}$ are:
$\mathbb{O} = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_3t_6t_{10}t_i$

belongs to. We have:

$$Nt_1t_3t_6t_{10}t_6t_6 = Nt_1t_3t_6t_{10} \in [13610]$$

After multiplying on the right by an element from each orbit we have concluded that there are no new double cosets. Then we checked and prove for those double cosets if they are equal to other existing double cosets.

We have completed the double coset enumeration of G, since the set of right cosets is closed under multiplication. Thus the index of N in G is 1120. We have concluded the following:

$$G = NeN \cup Nt_1N \cup Nt_1t_2N \cup Nt_1t_3N \cup Nt_1t_5N \cup Nt_1t_2t_1N$$

$$\cup Nt_1t_2t_3N \cup Nt_1t_2t_4N \cup Nt_1t_2t_5N \cup Nt_1t_2t_8N \cup Nt_1t_2t_9N$$

$$\cup Nt_1t_2t_{12}N \cup Nt_1t_3t_9N \cup Nt_1t_5t_{10}N \cup Nt_1t_5t_{11}N \cup Nt_1t_5t_{12}N$$

$$\cup Nt_1t_2t_3t_9N$$

where

$$G \cong \frac{2^{*13} : (13:4)}{(y^{-1}x^{-1}t_1^{x^6})^5, (x^2t_1)^5} \cong 2^{\bullet}S_z(8).$$

Therefore,

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(15)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(124)}|} + \frac{|N|}{|N^{(125)}|}$$

$$+ \frac{|N|}{|N^{(128)}|} + \frac{|N|}{|N^{(129)}|} + \frac{|N|}{|N^{(1212)}|} + \frac{|N|}{|N^{(139)}|} + \frac{|N|}{|N^{(1510)}|} + \frac{|N|}{|N^{(1511)}|} + \frac{|N|}{|N^{(1512)}|}$$

$$+ \frac{|N|}{|N^{(1239)}|}$$

and $|G| \leq (1+13+52+52+52+52+52+26+52+26+26+13+52+26+26+26+13) \times 52$

$|G| \leq 29120$.

The Cayley graph summarizes the information listed above.

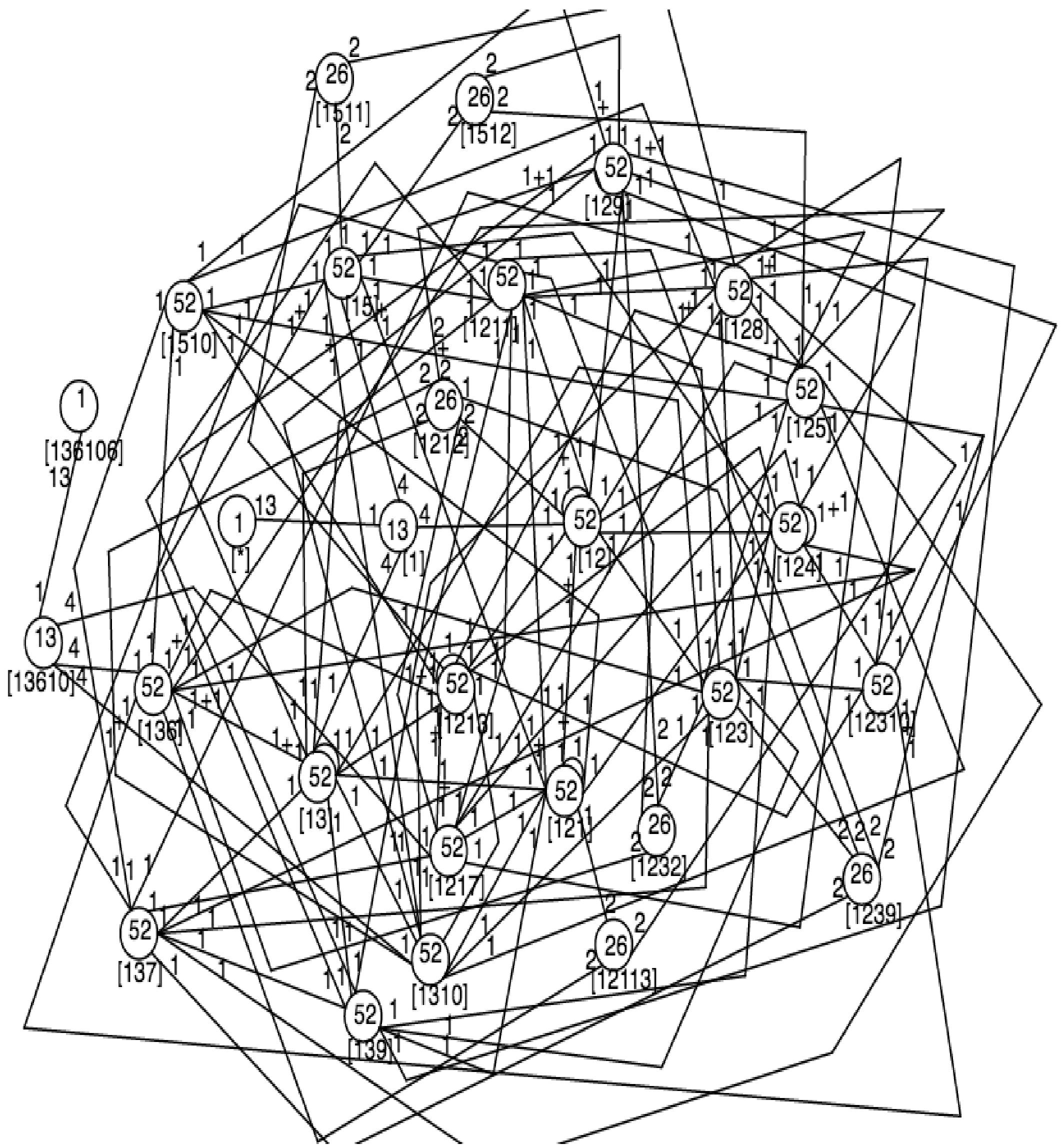

Figure 5.1: Cayley graph of $2^{\bullet}S_z(8)$ over $(13:4)$

5.1.2 Factoring $2^{\bullet}S_z(8)$ over $(13:4)$ by the Center $Z(G)$

Let $G = \frac{2^{*13}:(13:4)}{(y^{-1}x^{-1}t_1^{x^6})^5,(x^2t_1)^5} \cong 2^{\bullet}S_z(8)$. We want the group $G = 2^{*13} : D_{14}$ be factored by the relators $(y^{-1}x^{-1}t_1^{x^6})^5$, $(x^2t_1)^5$, and also factor by the center, to do so, we use the following loops in $MAGMA$. First we have the following finite presentation factored by first order relations.

```
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,y^x),
 ((y^-1*x^-1)*t^(x^6))^5,
 (x^2*t)^5>;
 #G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
 c:=Center(G1).1;
Order(c);
```

After checking the order of the center in $MAGMA$, we get that the order of G is 2. Now we use the Schreier System to convert the the center of G in terms of words, where $Center(G1)$ equals to c.

```
NN:=G;
 N:=G1;
 Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..58240]];
 for i in [2..58240] do
 P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
 if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
 if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
 if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
 end for;
 PP:=Id(N);
for k in [1..#P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;
 for i in [1..58240] do if ArrayP[i] eq c then Sch[i];
 end if; end for;
x^3 * t * x^-2 * t * y^-1 * t * y * t * y^2 * t
```

Therefore, we have converted the order on c into words which is the center of G or also known as $Z(G) =< x^3tx^{-2}ty^{-1}tyty^2t >$. Now we factor the finite presentation,G, by the first order relations and the center. We run the following loop in $MAGMA$ to verify that G and its center included we obtain the composition factors listed below:

```
>G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
>y^-1*x^-3*y*x^-2,t^2,(t,y^x),
>((y^-1*x^-1)*t^(x^6))^5,
>(x^2*t)^5,x^3*t*x^-2*t*y^-1*t*y*t*y^2*t>;
>f,G1,k:=CosetAction(G,sub<G|x,y>);
>CompositionFactors(G1);
 G
    |  2B(2, 8)                         = Sz(8)
    1
```

Hence, we have showed that G is factored by the relations $(y^{-1}x^{-1}t_1^{x^6})^5, (x^2t_1)^5$ and the center $Z(G) =< x^3tx^{-2}ty^{-1}tyty^2t >$.

5.1.3 Construction of $S_z(8)$ over $(13 : 4)$

Now that we know the center, $Z(G)$, of G we proceed on using the center, $x^3tx^{-2}ty^{-1}tyty^2t$, and the first order relations to factor G

Then,

$$G \cong \frac{2^{*13} : (13 : 4)}{(y^{-1}x^{-1}t_1^{x^6})^5, (x^2t_1)^5, x^3tx^{-2}ty^{-1}tyty^2t} \cong S_z(8).$$

Note $N = (13 : 4)$, where $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$ and

$y \sim (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7)$. Let $t \sim t_1$.

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N

NeN

First double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, the orbit of N on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ is: $\mathbb{O} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$. Let t_1 be our symmetric generator from this orbit $\mathbb{O}$ and find to which double coset Nt_1 belongs. Nt_1 will be a new double coset, denoted by [1], so thirteen symmetric generators will go to [1].

Nt_1N

In order to find how many single coset [1] contains, we must first find $N^{(1)}$. Then the number of single coset in [1] is equal to $\frac{|N|}{|N^{(1)}|}$. Now,

$$N^{(1)} = N^1\langle e, (2,9,13,6), (3,4,12,11)(5,7,10,8)\rangle$$

on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ so the number of the single cosets in Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{52}{4} = 13$. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:

$$\mathbb{O} = \{1\}, \{2,9,13,6\}, \{3,4,12,11\}, \text{and}\{5,7,10,8\}$$

We take t_1, t_2, t_3, and t_5 from each orbit, respectively, and find to which double coset Nt_1t_1, Nt_1t_2, Nt_1t_3, and Nt_1t_5 belong to. Now $Nt_1t_1 = N \in [*]$, so one element will go back to $[*]$. On the other hand, three symmetric generators will go to the new double cosets Nt_1t_2 denoted by [12], Nt_1t_3 denoted by [13], and Nt_1t_5 denoted by [15].

$\boldsymbol{Nt_1t_2N}$

Now Nt_1t_2N in N is a new double coset. We determine how many single cosets are in the double coset. Consider $N^{(12)} =< Id(N) >$. Then, $|N^{(12)}| = 1$ so the number of single cosets in $N^{(12)}$ is $\frac{|N|}{|N^{(12)}|} = \frac{52}{1} = 52$. The orbits of $N^{(12)}$ on {1,2,3,4,5,6,7,8,9,10,11,12,13} are: $\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and see which double coset $Nt_1t_2t_i$ belongs to. We have

$$\begin{aligned}
&Nt_1t_2t_1 \in [121]\\
&Nt_1t_2t_2 = Nt_1 \in [1]\\
&Nt_1t_2t_3 \in [123]\\
&Nt_1t_2t_4 \in [124]\\
&Nt_1t_2t_5 \in [125]\\
&t_1t_2t_6 = yt_4t_3t_4\\
\implies &Nt_1t_2t_6 = Nt_4t_3t_4 \in [121]\\
&t_1t_2t_7 = x^2y^{-1}t_1t_6\\
\implies &Nt_1t_2t_7 = Nt_1t_6 \in [12]
\end{aligned}$$

$$Nt_1t_2t_8 \in [128]$$
$$Nt_1t_2t_9 \in [129]$$
$$t_1t_2t_{10} = xyx^{-1}t_1t_9$$
$$\implies Nt_1t_2t_{10} = Nt_1t_9 \in [12]$$
$$t_1t_2t_{11} = yxy^{-1}t_{10}t_5t_3$$
$$\implies Nt_1t_2t_{11} = Nt_{10}t_5t_3 \in [125]$$
$$Nt_1t_2t_{12} \in [1212]$$
$$t_1t_2t_{13} = yx^{-1}t_4t_{12}t_4$$
$$\implies Nt_1t_2t_{13} = Nt_4t_{12}t_4 \in [121]$$

The new double cosets $Nt_1t_2t_1N$, $Nt_1t_2t_3N$, $Nt_1t_2t_4N$, $Nt_1t_2t_5N$, $Nt_1t_2t_8N$, $Nt_1t_2t_9N$, and $Nt_1t_2t_{12}N$ are denoted by $[121], [123], [124], [125], [128], [129]$, and $[1212]$ where $Nt_1t_2t_1$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_5$, $Nt_1t_2t_8$, $Nt_1t_2t_9$, and $Nt_1t_2t_{12}$ is its representative right coset.

Nt_1t_3N

Now Nt_1t_3N is a new double coset. However, $N^{(13)} = N^{13} = \langle Id(N) \rangle$. Only identity e will fix 1 and 3. Therefore the number of the single cosets in Nt_1t_3N is $\frac{|N|}{|N^{(13)}|} = \frac{52}{1} = 52$.

The orbits of $N^{(13)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_3t_i$ belongs to.
We have the following:

$$t_1t_3t_1 = yxt_4t_{12}t_{10}$$
$$\implies Nt_1t_3t_1 = Nt_4t_{12}t_{10} \in [125]$$
$$t_1t_3t_2 = y^{-1}xt_{13}t_1t_{13}$$
$$\implies Nt_1t_3t_2 = Nt_{13}t_1t_{13} \in [121]$$

$$
\begin{aligned}
Nt_1t_3t_3 &= Nt_1 \in [1] \\
t_1t_3t_4 &= yx^3t_4t_{12}t_3 \\
\implies Nt_1t_3t_4 &= Nt_4t_{12}t_3 \in [129] \\
t_1t_3t_5 &= yx^2y^{-1}t_9t_7 \\
\implies Nt_1t_3t_5 &= Nt_9t_7 \in [13] \\
t_1t_3t_6 &= y^{-1}xy^{-1}x^{-1}t_6t_{10} \\
\implies Nt_1t_3t_6 &= Nt_6t_{10} \in [15] \\
t_1t_3t_7 &= y^2x^2y^{-1}t_8t_{13}t_5 \\
\implies Nt_1t_3t_7 &= Nt_8t_{13}t_5 \in [123] \\
t_1t_3t_8 &= y^{-1}xy^{-1}x^{-1}t_{11}t_{10}t_7 \\
\implies Nt_1t_3t_8 &= Nt_{11}t_{10}t_7 \in [125] \\
Nt_1t_3t_9 &\in [139] \\
t_1t_3t_{10} &= y^{-1}xy^{-1}t_1t_3 \\
\implies Nt_1t_3t_{10} &= Nt_1t_3 \in [13] \\
t_1t_3t_{11} &= y^2x^3t_6t_7t_4 \\
\implies Nt_1t_3t_{11} &= Nt_6t_7t_4 \in [1212] \\
t_1t_3t_{12} &= y^{-1}xyt_1t_9t_1 \\
\implies Nt_1t_3t_{12} &= Nt_1t_9t_1 \in [121] \\
t_1t_3t_{13} &= y^2x^3t_9t_2 \\
\implies Nt_1t_3t_{13} &= Nt_9t_2 \in [15]
\end{aligned}
$$

The new double cosets $Nt_1t_3t_9N$ is denoted by [139] where $Nt_1t_3t_9$ is its representative right coset.

Nt_1t_5N

Also, Nt_1t_5N is a new double coset with $N^{(15)} = \langle Id(N)\rangle$. Only identity will fix 1 and 5. Hence, the number of single cosets in Nt_1t_5N is $\frac{|N|}{|N^{(15)}|} = \frac{52}{1} = 52$. The orbits of $N^{(15)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_i$ belongs to.

We have:

$$
\begin{aligned}
t_1t_5t_1 &= yxyx^{-1}t_9t_{11} \\
\implies Nt_1t_5t_1 &= Nt_9t_{11} \in [13] \\
t_1t_5t_2 &= y^{-1}xy^{-1}x^{-1}t_2t_4t_{10} \\
\implies Nt_1t_5t_2 &= Nt_2t_4t_{10} \in [139] \\
t_1t_5t_3 &= yxy^{-1}t_8t_3t_8 \\
\implies Nt_1t_5t_3 &= Nt_8t_3t_8 \in [121] \\
t_1t_5t_4 &= (yx)^2t_{11}t_{10}t_7 \\
\implies Nt_1t_5t_4 &= Nt_{11}t_{10}t_7 \in [125] \\
Nt_1t_5t_5 &= Nt_1 \in [1] \\
t_1t_5t_6 &= y^{-1}x^{-2}t_{13}t_5t_{13} \\
\implies Nt_1t_5t_6 &= Nt_{13}t_5t_{13} \in [121] \\
t_1t_5t_7 &= xy^{-1}t_7t_6t_5 \\
\implies Nt_1t_5t_7 &= Nt_7t_6t_5 \in [123] \\
t_1t_5t_8 &= y^{-1}xy^{-1}t_{13}t_{10} \\
\implies Nt_1t_5t_8 &= Nt_{13}t_{10} \in [13] \\
t_1t_5t_9 &= xyxt_7t_4t_8 \\
\implies Nt_1t_5t_9 &= Nt_7t_4t_8 \in [139] \\
Nt_1t_5t_{10} &\in [1510] \\
Nt_1t_5t_{11} &\in [1511] \\
Nt_1t_5t_{12} &\in [1512] \\
t_1t_5t_{13} &= y^2xy^{-1}t_3t_8t_5 \\
\implies Nt_1t_5t_{13} &= Nt_3t_8t_5 \in [124]
\end{aligned}
$$

The new double cosets $Nt_1t_5t_{10}N$, $Nt_1t_5t_{11}N$, and $Nt_1t_5t_{12}N$, are denoted by [1510], [1511], and [1512] where $Nt_1t_5t_{10}$, $Nt_1t_5t_{11}$, and $Nt_1t_5t_{12}$ is its representative right coset.

$\boldsymbol{Nt_1t_2t_1N}$

Consider $Nt_1t_2t_1N$ in N is a new double coset. We determined how many single cosets are on the double coset. $N^{(121)} = N^{121} = \langle e \rangle$. Now lets find the number of single cosets in $Nt_1t_2t_1N$ is $\frac{|N|}{|N^{(121)}|} = \frac{52}{1} = 52$. The orbits of $N(121)$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are
$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative call it t_i from each orbit and see which double cosets $Nt_1t_2t_1$ belongs to the following:

$$Nt_1t_2t_1t_1 = Nt_1t_2 \in [12]$$
$$t_1t_2t_1t_2 = xt_5t_{12}$$
$$\Longrightarrow Nt_1t_2t_1t_2 = Nt_5t_{12} \in [15]$$
$$t_1t_2t_1t_3 = yx^{-2}t_2t_4$$
$$\Longrightarrow Nt_1t_2t_1t_3 = Nt_2t_4 \in [13]$$
$$t_1t_2t_1t_4 = x^{-1}t_1t_{11}$$
$$\Longrightarrow Nt_1t_2t_1t_4 = Nt_1t_{11} \in [13]$$
$$t_1t_2t_1t_5 = y^2t_8t_{10}t_3$$
$$\Longrightarrow Nt_1t_2t_1t_5 = Nt_8t_{10}t_3 \in [139]$$
$$t_1t_2t_1t_6 = y^{-1}xy^{-1}t_{12}t_4t_1$$
$$\Longrightarrow Nt_1t_2t_1t_6 = Nt_{12}t_4t_1 \in [124]$$
$$t_1t_2t_1t_7 = x^2y^{-1}t_{12}t_4$$
$$\Longrightarrow Nt_1t_2t_1t_7 = Nt_{12}t_4 \in [12]$$
$$t_1t_2t_1t_8 = y^2x^3t_8t_4t_{12}$$
$$\Longrightarrow Nt_1t_2t_1t_8 = Nt_8t_4t_{12} \in [1510]$$
$$t_1t_2t_1t_9 = y^2t_1t_2t_1$$
$$\Longrightarrow Nt_1t_2t_1t_9 = Nt_1t_2t_1 \in [121]$$
$$t_1t_2t_1t_{10} = y^2xy{-}1x{-}1t_9t_2$$
$$\Longrightarrow Nt_1t_2t_1t_{10} = Nt_9t_2 \in [15]$$
$$t_1t_2t_1t_{11} = y^2x^{-2}t_9t_{10}t_{13}$$
$$\Longrightarrow Nt_1t_2t_1t_{11} = Nt_9t_{10}t_13 \in [125]$$

$$t_1t_2t_1t_{12} = xy^{-1}xt_4t_3$$
$$\implies Nt_1t_2t_1t_{12} = Nt_4t_3 \in [12]$$
$$t_1t_2t_1t_{13} = y^2xt_1t_{10}t_4$$
$$\implies Nt_1t_2t_1t_{13} = Nt_1t_{10}t_4 \in [1511]$$

$Nt_1t_2t_3N$

Consider $Nt_1t_2t_3N$ in N is a new double coset. However, $N^{(123)} = N^{123} = \langle Id(N)\rangle$. Only identity fix 1,2 and 3. Hence, the number of single cosets in $Nt_1t_2t_3$ is $\frac{|N|}{|N^{(123)}|} = \frac{52}{1} = 52$. The orbits of $N^{(123)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are: $\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative $t_i$from each orbit and check which double coset $Nt_1t_2t_3t_i$ belongs to.

$$t_1t_2t_3t_1 = y^2x^2y^{-1}t_7t_3$$
$$\implies Nt_1t_2t_3t_1 = Nt_7t_3 \in [15]$$
$$t_1t_2t_3t_2 = y^2x^3t_{11}t_5t_1$$
$$\implies Nt_1t_2t_3t_2 = Nt_{11}t_5t_1 \in [1512]$$
$$Nt_1t_2t_3t_3 = Nt_1t_2 \in [12]$$
$$t_1t_2t_3t_4 = (y,x^{-1})t_7t_6t_5$$
$$\implies Nt_1t_2t_3t_4 = Nt_7t_6t_5 \in [123]$$
$$t_1t_2t_3t_5 = y^2xyxt_5t_6t_{12}$$
$$\implies Nt_1t_2t_3t_5 = Nt_5t_6t_{12} \in [128]$$
$$t_1t_2t_3t_6 = y^2xyt_{10}t_{13}$$
$$\implies Nt_1t_2t_3t_6 = Nt_{10}t_{13} \in [13]$$
$$t_1t_2t_3t_7 = xyxt_1t_{11}t_2$$
$$\implies Nt_1t_2t_3t_7 = Nt_1t_{11}t_2 \in [139]$$
$$t_1t_2t_3t_8 = y^{-1}x^2t_6t_2t_{10}$$
$$\implies Nt_1t_2t_3t_8 = Nt_6t_2t_{10} \in [1510]$$

$$
\begin{aligned}
& Nt_1t_2t_3t_9 \in [1239] \\
& t_1t_2t_3t_{10} = (y^{-1}x)^2 t_4t_7t_3 \\
\implies & Nt_1t_2t_3t_{10} = Nt_4t_7t_3 \in [139] \\
& t_1t_2t_3t_{11} = y^{-1}x^{-1}t_6t_1t_{12} \\
\implies & Nt_1t_2t_3t_{11} = Nt_6t_1t_{12} \in [125] \\
& t_1t_2t_3t_{12} = y^2t_7t_{12}t_1 \\
\implies & Nt_1t_2t_3t_{12} = Nt_7t_{12}t_1 \in [125] \\
& t_1t_2t_3t_{13} = y^2xy^{-1}xt_4t_{12}t_2 \\
\implies & Nt_1t_2t_3t_{13} = Nt_4t_{12}t_2 \in [124]
\end{aligned}
$$

The new double cosets $Nt_1t_2t_3t_9N$ is denoted by [1239] where $Nt_1t_2t_3t_9$ is its representative right coset.

$Nt_1t_2t_4N$

Now $Nt_1t_2t_4N$ in N is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(124)} = N^{124} = \langle Id(N)\rangle$. But $Nt_1t_2t_4$ is not distinct. Now $Nt_2t_1t_{12} \in [124]$ since $(1,2)(3,13)(4,12)(5,11)(6,10)(7,9) \in N$ and
$N(t_1t_2t_4)^{(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)} = Nt_2t_1t_{12}$.
Thus, $(1,2)(3,13)(4,12)(5,11)(6,10)(7,9) \in N^{(124)}$. We conclude:

$$N^{(124)} \geq \langle (1,2)(3,13)(4,12)(5,11)(6,10)(7,9)\rangle.$$

Hence $|N^{(124)}| = 2$ so the number of single cosets in $N^{(124)}$ is $\frac{|N|}{|N^{(124)}|} = \frac{52}{2} = 26$. The orbits of $N^{(124)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{8\}, \{1,2\}, \{3,13\}, \{4,12\}, \{5,11\}, \{6,10\}, \{7,9\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_4t_i$ belongs to. We have:

$$
\begin{aligned}
& t_1t_2t_4t_8 = xyx^{-1}t_9t_{10}t_4 \\
\implies & Nt_1t_2t_4t_8 = Nt_9t_{10}t_4 \in [129]
\end{aligned}
$$

$$t_1t_2t_4t_1 = x^2y^{-1}t_{13}t_2t_8$$
$$\Longrightarrow Nt_1t_2t_4t_1 = Nt_{13}t_2t_8 \in [139]$$
$$t_1t_2t_4t_3 = y^2xyxt_{11}t_4$$
$$\Longrightarrow Nt_1t_2t_4t_3 = Nt_{11}t_4 \in [15]$$
$$Nt_1t_2t_4t_4 = Nt_1t_2 \in [12]$$
$$t_1t_2t_4t_5 = (y^{-1}x)^2t_4t_{12}t_4$$
$$\Longrightarrow Nt_1t_2t_4t_5 = Nt_4t_{12}t_4 \in [121]$$
$$t_1t_2t_4t_6 = (yxy^{-1})^2t_{11}t_{10}t_8$$
$$\Longrightarrow Nt_1t_2t_4t_6 = Nt_{10}t_{11}t_8 \in [124]$$
$$t_1t_2t_4t_7 = xy^{-1}x^{-1}t_{12}t_4t_9$$
$$\Longrightarrow Nt_1t_2t_4t_7 = Nt_{12}t_4t_9 \in [123]$$

$Nt_1t_2t_5N$

Consider $Nt_1t_2t_5N$ in N is a new double coset. However, $N^{(125)} = N^{125} = \langle Id(N)\rangle$. Only identity fix 1,2 and 5. The number of single cosets in $Nt_1t_2t_5$ is $\frac{|N|}{|N^{(125)}|} = \frac{52}{1} = 52$. The orbits of $N^{(125)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_2t_5t_i$ belongs to.

$$t_1t_2t_5t_1 = yxt_1t_8t_4$$
$$\Longrightarrow Nt_1t_2t_5t_1 = Nt_1t_8t_4 \in [1512]$$
$$t_1t_2t_5t_2 = yxyx^{-1}t_5t_{13}t_8$$
$$\Longrightarrow Nt_1t_2t_5t_2 = Nt_5t_{13}t_8 \in [123]$$
$$t_1t_2t_5t_3 = yxyt_6t_7t_6$$
$$\Longrightarrow Nt_1t_2t_5t_3 = Nt_6t_7t_6 \in [121]$$
$$t_1t_2t_5t_4 = yxyx^{-1}t_{11}t_9$$
$$\Longrightarrow Nt_1t_2t_5t_4 = Nt_{11}t_9 \in [13]$$
$$Nt_1t_2t_5t_5 = Nt_1t_2 \in [12]$$

$$t_1t_2t_5t_6 = xt_8t_{13}$$
$$\implies Nt_1t_2t_5t_6 = Nt_8t_{13} \in [12]$$
$$t_1t_2t_5t_7 = y^2x^{-1}t_{11}t_1t_{10}$$
$$\implies Nt_1t_2t_5t_7 = Nt_{11}t_1t_{10} \in [139]$$
$$t_1t_2t_5t_8 = y^2x^2t_{11}t_7$$
$$\implies Nt_1t_2t_5t_8 = Nt_{11}t_7 \in [15]$$
$$t_1t_2t_5t_9 = y^2x^2t_1t_2t_5$$
$$\implies Nt_1t_2t_5t_9 = Nt_1t_2t_5 \in [125]$$
$$t_1t_2t_5t_{10} = yx^3t_5t_4t_{11}$$
$$\implies Nt_1t_2t_5t_{10} = Nt_5t_4t_{11} \in [128]$$
$$t_1t_2t_5t_{11} = xy^{-1}t_3t_8t_{13}t_4$$
$$\implies Nt_1t_2t_5t_{11} = Nt_3t_8t_{13}t_4 \in [1239]$$
$$t_1t_2t_5t_{12} = y^{-1}x^{-1}t_{12}t_9$$
$$\implies Nt_1t_2t_5t_{12} = Nt_{12}t_9 \in [13]$$
$$t_1t_2t_5t_{13} = xyxt_2t_7t_{12}$$
$$\implies Nt_1t_2t_5t_{13} = Nt_2t_7t_{12} \in [123]$$

$Nt_1t_2t_8N$

Also $Nt_1t_2t_8N$ in N is a new double coset. With $N^{(128)} = N^{128} = \langle Id(N) \rangle$. But $Nt_1t_2t_8N$ is not distinct. Now $Nt_{10}t_9t_3 \in [128]$ since $(1,10)(2,9)(3,8)(4,7)$ $(5,6)(11,13) \in N$ and $N(t_1t_2t_8)^{(1,10)(2,9)(3,8)(4,7)(5,6)(11,13)} = Nt_{10}t_9t_3$. Thus, $(1,10)(2,9)(3,8)(4,7)(5,6)(11,13) \in N^{(128)}$. We conclude:

$$N^{(128)} \geq \langle (1,10)(2,9)(3,8)(4,7)(5,6)(11,13) \rangle.$$

Hence $|N^{(128)}| = 2$ so the number of single cosets in $N^{(128)}$ is $\frac{|N|}{|N^{(128)}|} = \frac{52}{2} = 26$. The orbits of $N^{(128)}$ are:

$\mathbb{O} = \{12\}, \{1,10\}, \{2,9\}, \{3,8\}, \{4,7\}, \{5,6\}, \{11,13\}$.

Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_8t_i$ belongs to. We have:

$$t_1t_2t_8t_{12} = y^2x^3t_8t_{12}t_4$$
$$\implies Nt_1t_2t_8t_{12} = Nt_8t_{12}t_4 \in [1510]$$
$$t_1t_2t_8t_1 = xyt_{10}t_{11}t_{12}$$
$$\implies Nt_1t_2t_8t_1 = Nt_{10}t_{11}t_{12} \in [123]$$
$$t_1t_2t_8t_2 = y^{-1}t_6t_7t_{10}$$
$$\implies Nt_1t_2t_8t_2 = Nt_6t_7t_{10} \in [125]$$
$$Nt_1t_2t_8t_8 = Nt_1t_2 \in [12]$$
$$t_1t_2t_8t_4 = y^2x^3t_1t_8t_{12}$$
$$\implies Nt_1t_2t_8t_4 = Nt_1t_8t_{12} \in [1511]$$
$$t_1t_2t_8t_5 = (y^{-1}, x^{-1})t_2t_{12}t_3$$
$$\implies Nt_1t_2t_8t_5 = Nt_2t_{12}t_3 \in [139]$$
$$t_1t_2t_8t_{11} = x^{-2}t_5t_3t_{10}$$
$$\implies Nt_1t_2t_8t_{11} = Nt_5t_3t_{10} \in [139]$$

$\boldsymbol{Nt_1t_2t_9N}$

Now $Nt_1t_2t_9N$ in N is a new double coset with $N^{(129)} = N^{129} = \langle Id(N)\rangle$. But $Nt_1t_2t_9N$ is not distinct since $Nt_{12}t_{11}t_4 \in [129]$ and $(1,12)(2,11)(3,10)(4,9)(5,8)$ $(6,7) \in N$.Then $N(t_1t_2t_9)^{(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)} = Nt_{12}t_{11}t_4$.
Thus, $(1,12)(2,11)(3,10)(4,9)(5,8)(6,7) \in N^{(129)}$. We conclude:

$$N^{(129)} \geq \langle (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)\rangle.$$

Hence $|N^{(129)}| = 2$ so the number of single cosets in $N^{(129)}$ is $\frac{|N|}{|N^{(129)}|} = \frac{52}{2} = 26$.
The orbits of $N^{(129)}$ are: $\mathbb{O} = \{13\}, \{1,12\}, \{2,11\}, \{3,10\}, \{4,9\}, \{5,8\}, \{6,7\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_9t_i$ belongs to.

$$t_1t_2t_9t_{13} = y^2xyt_7t_6t_4$$
$$\implies Nt_1t_2t_9t_{13} = Nt_7t_6t_4 \in [124]$$
$$t_1t_2t_9t_1 = y^{-1}xt_{12}t_9$$
$$\implies Nt_1t_2t_9t_1 = Nt_{12}t_9 \in [13]$$
$$t_1t_2t_9t_2 = y^2x^3t_2t_3t_{13}$$
$$\implies Nt_1t_2t_9t_2 = Nt_2t_3t_{13} \in [1212]$$
$$t_1t_2t_9t_3 = yt_{11}t_{13}t_6$$
$$\implies Nt_1t_2t_9t_3 = Nt_{11}t_{13}t_6 \in [139]$$
$$t_1t_2t_9t_5 = y^2x^{-1}t_6t_{10}t_4$$
$$\implies Nt_1t_2t_9t_5 = Nt_6t_{10}t_4 \in [1512]$$
$$t_1t_2t_9t_6 = yxy^{-1}t_8t_1t_2$$
$$\implies Nt_1t_2t_9t_6 = Nt_8t_1t_2 \in [1510]$$
$$Nt_1t_2t_9t_9 = Nt_1t_2 \in [12]$$

$\boldsymbol{Nt_1t_2t_{12}N}$

Consider $Nt_1t_2t_{12}N$ in N is a new double coset with $N^{(1212)} = N^{1212} = \langle e \rangle$. But $Nt_1t_2t_{12}N$ is not distinct since $Nt_{10}t_2t_{13} \in [1212]$ and $(1,10,3,7)(4,12,13,5)$ $(6,9,11,8) \in N$.Then $N(t_1t_2t_{12})^{(1,10,3,7)(4,12,13,5)(6,9,11,8)} = Nt_{10}t_2t_{13}$.
Thus, $(1,10,3,7)(4,12,13,5)(6,9,11,8) \in N^{(1212)}$. We conclude:

$$N^{(1212)} \geq \langle (1,10,3,7)(4,12,13,5)(6,9,11,8) \rangle.$$

Hence $|N^{(1212)}| = 4$ so the number of single cosets in $N^{(1212)}$ is $\frac{|N|}{|N^{(1212)}|} = \frac{52}{4} = 13$.
The orbits of $N^{(1212)}$ are: $\mathbb{O} = \{2\}, \{1,10,3,7\}, \{4,12,13,5\}, \{6,9,11,8\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_{12}t_i$ belongs

to.

$$\begin{aligned}
t_1t_2t_{12}t_2 &= y^{-1}x^2y^{-1}t_{13}t_5t_{10}t_1 \\
\implies Nt_1t_2t_{12}t_2 &= Nt_{13}t_5t_{10}t_1 \in [1239] \\
t_1t_2t_{12}t_1 &= y^2x^2t_{11}t_{10}t_3 \\
\implies Nt_1t_2t_{12}t_1 &= Nt_{11}t_{10}t_3 \in [129] \\
Nt_1t_2t_{12}t_{12} &= Nt_1t_2 \in [12] \\
t_1t_2t_{12}t_6 &= (y^{-1}x)^2t_9t_{11} \\
\implies Nt_1t_2t_{12}t_6 &= Nt_9t_{11} \in [13]
\end{aligned}$$

$\boldsymbol{Nt_1t_3t_9N}$

From above $Nt_1t_3t_9N$ in N is a new double coset with $N^{(139)} = N^{139} = \langle e \rangle$. Now we are going to check how many new single cosets are in the double cosets in $Nt_1t_3t_9$ is $\frac{|N|}{|N^{(139)}|} = \frac{52}{1} = 52$. The orbits of $N^{(139)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are:

$\mathbb{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}$. Take a representative t_i from each orbit and check which double coset $Nt_1t_3t_9t_i$
belongs to.

$$\begin{aligned}
t_1t_3t_9t_1 &= y^2x^2t_{13}t_4 \\
\implies Nt_1t_3t_9t_1 &= Nt_{13}t_4 \in [15] \\
t_1t_3t_9t_2 &= y^2xy^{-1}x^{-1}t_2t_3t_5 \\
\implies Nt_1t_3t_9t_2 &= Nt_2t_3t_5 \in [124] \\
t_1t_3t_9t_3 &= x^3t_5t_3t_{10} \\
\implies Nt_1t_3t_9t_3 &= Nt_5t_3t_{10} \in [139] \\
t_1t_3t_9t_4 &= y^{-1}x^3t_5t_{11} \\
\implies Nt_1t_3t_9t_4 &= Nt_5t_{11} \in [15] \\
t_1t_3t_9t_5 &= y^{-1}xy^{-1}t_{12}t_4t_9 \\
\implies Nt_1t_3t_9t_5 &= Nt_{12}t_4t_9 \in [123]
\end{aligned}$$

$$
\begin{aligned}
t_1t_3t_9t_6 &= y^{-1}x^2t_2t_1t_7 \\
\implies Nt_1t_3t_9t_6 &= Nt_2t_1t_7 \in [129] \\
t_1t_3t_9t_7 &= y^{-1}xy^{-1}x^{-1}t_3t_8t_{10} \\
\implies Nt_1t_3t_9t_7 &= Nt_3t_8t_{10} \in [125] \\
t_1t_3t_9t_8 &= (y^{-1},x^{-1})t_9t_{10}t_3 \\
\implies Nt_1t_3t_9t_8 &= Nt_9t_{10}t_3 \in [128] \\
Nt_1t_3t_9t_9 &= Nt_1t_3 \in [13] \\
t_1t_3t_9t_{10} &= xy^{-1}xt_1t_9t_4 \\
\implies Nt_1t_3t_9t_{10} &= Nt_1t_9t_5 \in [123] \\
t_1t_3t_9t_{11} &= y^2x^{-1}t_7t_8t_7 \\
\implies Nt_1t_3t_9t_{11} &= Nt_7t_8t_7 \in [121] \\
t_1t_3t_9t_{12} &= (y,x^{-1})t_6t_1t_{10} \\
\implies Nt_1t_3t_9t_{12} &= Nt_6t_1t_{10} \in [128] \\
t_1t_3t_9t_{13} &= yxt_{13}t_6t_2 \\
\implies Nt_1t_3t_9t_{13} &= Nt_{13}t_6t_2 \in [1511]
\end{aligned}
$$

$\boldsymbol{Nt_1t_5t_{10}N}$

Now $Nt_1t_5t_{10}N$ in N is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(1510)} = N^{1510} = \langle Id(N)\rangle$. But $Nt_1t_5t_{10}$ is not distinct. Now $Nt_9t_5t_{13} \in [1210]$ since $(1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \in N$ and $N(t_1t_2t_{10})^{(1,9)(2,8)(3,7)(4,6)(10,13)(11,12)} = Nt_9t_5t_{13}$.
Thus, $(1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \in N^{(1210)}$. We conclude:

$$N^{(1210)} \geq \langle (1,9)(2,8)(3,7)(4,6)(10,13)(11,12)\rangle.$$

Hence $|N^{(1210)}| = 2$ so the number of single cosets in $N^{(1210)}$ is $\frac{|N|}{|N^{(1210)}|} = \frac{52}{2} = 26$. The orbits of $N^{(1210)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ are:
$\mathbb{O} = \{5\},\{1,9\},\{2,8\},\{3,7\},\{4,6\},\{10,13\},\{11,12\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_{10}t_i$ belongs

to. We have:

$$t_1t_5t_{10}t_5 = y^2x^2t_7t_8t_1$$
$$\implies Nt_1t_5t_{10}t_5 = Nt_7t_8t_1 \in [128]$$
$$t_1t_5t_{10}t_1 = y^2x^2t_8t_7t_8$$
$$\implies Nt_1t_5t_{10}t_1 = Nt_8t_7t_8 \in [121]$$
$$t_1t_5t_{10}t_8 = y^2x^{-2}t_{12}t_7t_2t_{11}$$
$$\implies Nt_1t_5t_{10}t_8 = Nt_{12}t_7t_2t_{11} \in [1239]$$
$$t_1t_5t_{10}t_3 = yxt_6t_2t_9$$
$$\implies Nt_1t_5t_{10}t_3 = Nt_6t_2t_9 \in [1511]$$
$$t_1t_5t_{10}t_4 = yxy^{-1}xt_8t_3t_7$$
$$\implies Nt_1t_5t_{10}t_4 = Nt_8t_3t_7 \in [129]$$
$$Nt_1t_5t_{10}t_{10} = Nt_1t_5 \in [15]$$
$$t_1t_5t_{10}t_{11} = xyxt_4t_5t_6$$
$$\implies Nt_1t_5t_{10}t_{11} = Nt_4t_5t_6 \in [123]$$

$Nt_1t_5t_{11}N$

Consider $Nt_1t_5t_{11}N$ in N is a new double coset with $N^{(1511)} = N^{1511} = \langle e \rangle$. But $Nt_1t_5t_{11}N$ is not distinct since $Nt_6t_2t_9 \in [1511]$ and $(1,6)(2,5)(3,4)(7,13)(8,12)$ $(9,11) \in N$.Then $N(t_1t_5t_{11})^{(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)} = Nt_6t_2t_9$.
Thus, $N^{(1511)} \geq \langle (1,6)(2,5)(3,4)(7,13)(8,12)(9,11) \rangle$. Hence $|N^{(1511)}| = 2$ so the number of single cosets in $N^{(1511)}$ is $\frac{|N|}{|N^{(1511)}|} = \frac{52}{2} = 26$.
The orbits of $N^{(1511)}$ are: $\mathbb{O} = \{10\}, \{1,6\}\{2,5\}, \{3,4\}\{7,13\}, \{8,12\}\{9,11\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_{11}t_i$ belongs to.

$$t_1t_5t_{11}t_{10} = y^{-1}xy^{-1}x^{-1}t_4t_{11}t_2$$
$$\implies Nt_1t_5t_{11}t_{10} = Nt_4t_{11}t_2 \in [1511]$$

$$t_1t_5t_{11}t_1 = x^2y^{-1}t_6t_3t_7$$
$$\implies Nt_1t_5t_{11}t_1 = Nt_6t_3t_7 \in [139]$$
$$t_1t_5t_{11}t_2 = y^2x^3t_1t_{13}t_1$$
$$\implies Nt_1t_5t_{11}t_2 = Nt_1t_{13}t_1 \in [121]$$
$$t_1t_5t_{11}t_3 = y^2xyxt_9t_5t_{13}$$
$$\implies Nt_1t_5t_{11}t_3 = Nt_9t_5t_{13} \in [1510]$$
$$t_1t_5t_{11}t_7 = y^{-1}x^2t_{11}t_5t_1$$
$$\implies Nt_1t_5t_{11}t_7 = Nt_{11}t_5t_1 \in [1512]$$
$$t_1t_5t_{11}t_8 = y^2x^3t_6t_{11}t_2$$
$$\implies Nt_1t_5t_{11}t_8 = Nt_6t_{11}t_2 \in [128]$$
$$Nt_1t_5t_{11}t_{11} = Nt_1t_5 \in [15]$$

$\boldsymbol{Nt_1t_5t_{12}N}$

Now $Nt_1t_5t_{12}N$ in N is a new double coset with $N^{(1512)} = N^{1512} = \langle Id(N)\rangle$. But $Nt_1t_5t_{12}N$ is not distinct since $Nt_3t_{12}t_5 \in [129]$ and $(1,12)(2,11)(3,10)(4,9)$ $(5,8)(6,7) \in N$.Then $N(t_1t_2t_9)^{(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)} = Nt_3t_{12}t_5$. Thus, $(1,12)(2,11)(3,10)(4,9)(5,8)(6,7) \in N^{(1512)}$. We conclude:

$$N^{(1512)} \geq \langle (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)\rangle.$$

Hence $|N^{(1512)}| = 2$ so the number of single cosets in $N^{(1512)}$ is $\frac{|N|}{|N^{(1512)}|} = \frac{52}{2} = 26$. The orbits of $N^{(1512)}$ are: $\mathbb{O} = \{2\}, \{1,3\}, \{4,13\}, \{5,12\}, \{6,11\}, \{7,10\}, \{8,9\}$. Take a representative t_i from each orbit and see which double cosets $Nt_1t_5t_{12}t_i$ belongs to.

$$t_1t_5t_{12}t_2 = y^2x^2t_{10}t_4t_{13}$$
$$\implies Nt_1t_5t_{12}t_2 = Nt_{10}t_4t_{13} \in [1512]$$

$$t_1t_5t_{12}t_1 = y^{-1}xt_1t_9t_7$$
$$\implies Nt_1t_5t_{12}t_1 = Nt_1t_9t_7 \in [125]$$
$$t_1t_5t_{12}t_4 = y^2xt_8t_7t_{13}$$
$$\implies Nt_1t_5t_{12}t_4 = Nt_8t_7t_{13} \in [129]$$
$$Nt_1t_5t_{12}t_{12} = Nt_1t_5 \in [15]$$
$$t_1t_5t_{12}t_6 = y^2xt_1t_5t_{12}$$
$$\implies Nt_1t_5t_{12}t_6 = Nt_1t_5t_{12} \in [1512]$$
$$t_1t_5t_{12}t_7 = (y^{-1}x)^2t_{12}t_7t_2$$
$$\implies Nt_1t_5t_{12}t_7 = Nt_{12}t_7t_2 \in [123]$$
$$t_1t_5t_{12}t_8 = yt_{12}t_5t_1$$
$$\implies Nt_1t_5t_{12}t_8 = Nt_{12}t_5t_1 \in [1511]$$

$\boldsymbol{Nt_1t_2t_3t_9N}$

Also $Nt_1t_2t_3t_9N$ in N is a new double coset. With $N^{(1239)} = N^{1239} = \langle Id(N)\rangle$. But $Nt_1t_2t_3t_9N$ is not distinct. Now $Nt_6t_1t_9t_5 \in [1239]$ since $(1,6,7,2)(3,9,5,12)(8,10,13,11) \in N$ and $N(t_1t_2t_3t_9)^{(1,6,7,2)(3,9,5,12)(8,10,13,11)} = Nt_6t_1t_9t_5$. Thus, $(1,6,7,2)(3,9,5,12)(8,10,13,11) \in N^{(1239)}$. We conclude:

$$N^{(1239)} \geq \langle(1,6,7,2)(3,9,5,12)(8,10,13,11)\rangle.$$

Hence $|N^{(1239)}| = 4$ so the number of single cosets in $N^{(1239)}$ is $\frac{|N|}{|N^{(1239)}|} = \frac{52}{4} = 13$. The orbits of $N^{(1239)}$ are:
$\mathbb{O} = \{4\}, \{1,7,6,2\}, \{3,5,9,12\}, \{8,13,10,11\}$.
Take a representative t_i from each orbit and see which double cosets $Nt_1t_2t_3t_9t_i$ belongs to. We have:

$$t_1t_2t_3t_9t_4 = y^2x^2t_5t_4t_7$$
$$\implies Nt_1t_2t_3t_9t_4 = Nt_5t_4t_7 \in [1212]$$
$$t_1t_2t_3t_9t_1 = y^2xt_{10}t_3t_4$$
$$\implies Nt_1t_2t_3t_9t_1 = Nt_{10}t_3t_4 \in [1510]$$

$$Nt_1t_2t_3t_9t_9 = Nt_1t_2t_3 \in [123]$$
$$t_1t_2t_3t_9t_8 = yx^{-1}t_{10}t_2t_4$$
$$\implies Nt_1t_2t_3t_9t_8 = Nt_{10}t_2t_4 \in [125]$$

After multiplying on the right by an element from each orbit we have concluded that there are no new double cosets. Then we checked and prove for those double cosets if they are equal to other existing double cosets.
We have completed the double coset enumeration of G, since the set of right cosets is closed under multiplication. Thus the index of N in G is 560. We have concluded the following:

$$G = NeN \cup Nt_1N \cup Nt_1t_2N \cup Nt_1t_3N \cup Nt_1t_5N \cup Nt_1t_2t_1N$$
$$\cup Nt_1t_2t_3N \cup Nt_1t_2t_4N \cup Nt_1t_2t_5N \cup Nt_1t_2t_8N \cup Nt_1t_2t_9N$$
$$\cup Nt_1t_2t_{12}N \cup Nt_1t_3t_9N \cup Nt_1t_5t_{10}N \cup Nt_1t_5t_{11}N \cup Nt_1t_5t_{12}N$$
$$\cup Nt_1t_2t_3t_9N$$

where
$$G \cong \frac{2^{*13} : (13:4)}{((y^{-1}x^{-1}t_1^{x^6})^5, (x^2t_1)^5} \cong S_z(8).$$

Therefore,

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(15)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(124)}|} + \frac{|N|}{|N^{(125)}|}$$
$$+ \frac{|N|}{|N^{(128)}|} + \frac{|N|}{|N^{(129)}|} + \frac{|N|}{|N^{(1212)}|} + \frac{|N|}{|N^{(139)}|} + \frac{|N|}{|N^{(1510)}|} + \frac{|N|}{|N^{(1511)}|} + \frac{|N|}{|N^{(1512)}|}$$
$$+ \frac{|N|}{|N^{(1239)}|}$$

and

$|G| \leq (1 + 13 + 52 + 52 + 52 + 52 + 52 + 26 + 52 + 26 + 26 + 13 + 52 + 26 + 26 + 26 + 13) \times 52$
$|G| \leq 29120.$

The Cayley graph summarizes the information listed above.

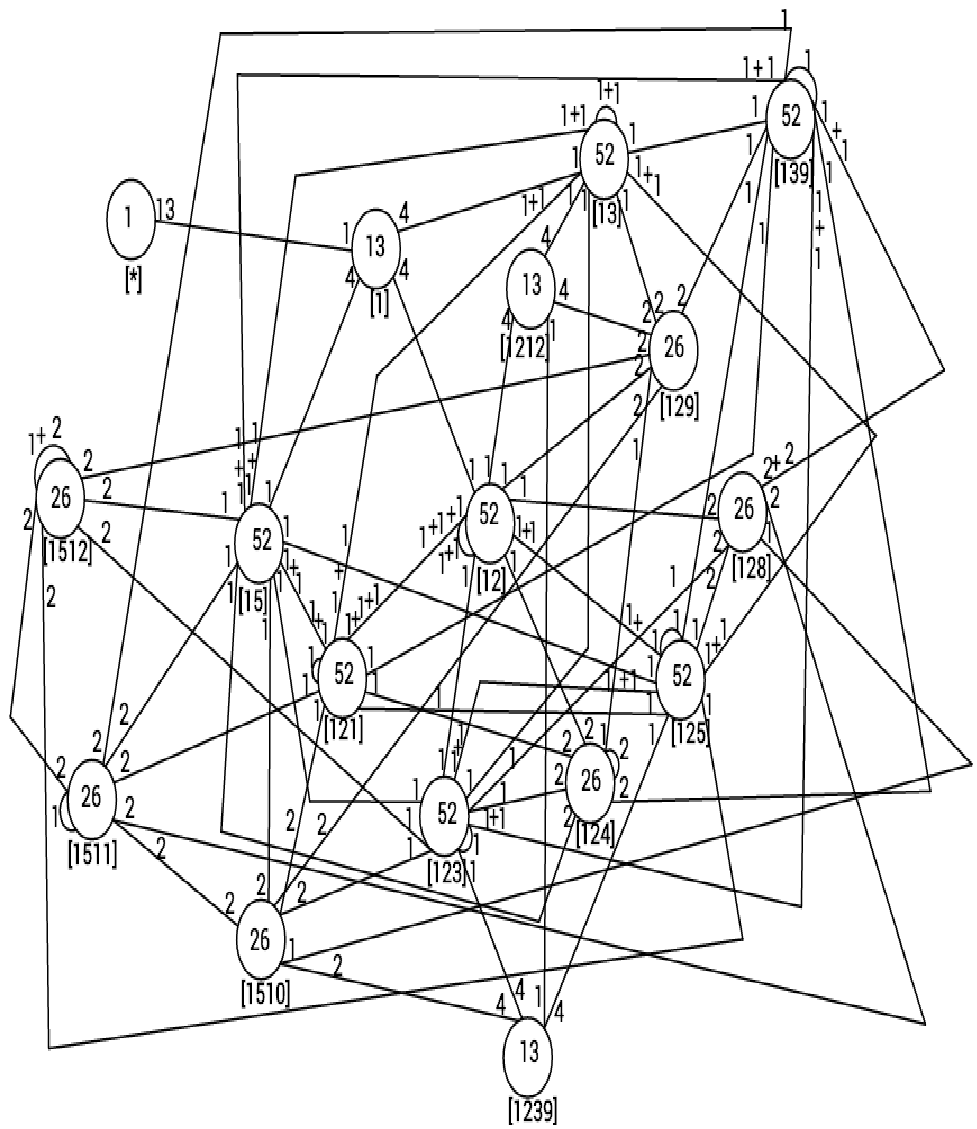

Figure 5.2: Cayley graph of $S_z(8)$ over $(13:4)$

Chapter 6

Double Coset Enumeration over a Maximal Subgroup

In this chapter we will construct a double coset enumeration over a maximal subgroup with a progenitor factor by additional relations.

6.1 Construction of $2 \times PGL_2(27)$ over $M = 2^{\bullet}(13:2)$

Definition 6.1. *Let G be a group and $H \leq G$. H is a* **maximal subgroup** *of G if there is no normal subgroup $N \leq G$ such that $H < N < G$. [Rot95]*

6.1.1 Double Coset Enumeration of G

In order to construct a double coset enumeration we have to consider on obtaining the homomorphic image by factoring the progenitor $2^{*13} : (13:2)$ by the relations $((x^4)tt^x)^3$, and $((x^6)tt^x)^2$ denoted by:

$$\langle x, y, t | y^2, (x^{-1}y)^2, x^{-13},$$

$$t^2,$$

$$(t, yx^2), ((x^4)tt^x)^3, ((x^6)tt^x)^2\rangle$$

and $N \cong (13:2) =$

$$\langle x, y | y^2, (x^{-1}y)^2, x^{-13}.$$

Then,

$$G \cong \frac{2^{*13} : (13:2)}{((x^4)tt^x)^3, ((x^6)tt^x)^2} \cong 2 \times PGL_2(27).$$

Let $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13)$ and $y \sim (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$, where $t \sim t_1$ and $\pi = x^4$.

The first relation, $((x^4)tt^x)^3$ or $(\pi t_1 t_2)^3$, is expand as follow:

$$\begin{aligned}
(\pi t_1 t_2)^3 &= 1 \\
\pi^3 (t_1 t_2)^{\pi^2} (t_1 t_2)^{\pi} t_1 t_2 &= \\
\pi^3 t_1^{\pi^2} t_2^{\pi^2} t_1^{\pi} t_2^{\pi} t_1 t_2 &= \\
\pi^3 t_1^{(1,9,4,\ldots)} t_2^{(1,9,4,12,7,2,10\ldots)} t_1^{(1,5,9,13,\ldots)} t_2^{(1,5,9,\ldots 2,6,10)} t_1 t_2 &= \\
\pi^3 t_9 t_{10} t_5 t_6 t_1 t_2 &= \\
(1,13,12,11,10,9,8,7,6,5,4,3,2) t_9 t_{10} t_5 t_6 t_1 t_2 &= 1 \\
(1,13,12,11,10,9,8,7,6,5,4,3,2) t_9 t_{10} t_5 &= t_2 t_1 t_6
\end{aligned}$$

Expand second relation $(\beta t_1 t_1^x)^2$ where $\beta = x^6$:

$$\begin{aligned}
(\beta t_1 t_1^x)^2 &= e \\
(\beta t_1 t_2)^2 &= \\
\beta^2 (t_1 t_2)^{\beta} t_1 t_2 &= \\
\beta^2 t_1^{\beta} t_2^{\beta} t_1 t_2 &= e \\
(1,13,12,11,10,9,8,7,6,5,4,3,2) t_7 t_8 t_1 t_2 &= e \\
(1,13,12,11,10,9,8,7,6,5,4,3,2) t_7 t_8 &= t_2 t_1
\end{aligned}$$

There is one more thing to consider, let M be the maximal subgroup generated by the control group $N = (13:2)$ and $ytx^4txtyt = t_7t_{11}t_{12}t_1$ since

$$\begin{aligned}
&ytx^4txtyt\\
&ytx^4txyy^{-1}tyt\\
&ytx^4txyt^yt\\
&ytx^4(xy)(xy)^{-1}txyt^yt\\
&ytx^5yt^{(xy)}t^yt\\
&y(x^5y)(x^5y)^{-1}t(x^5y)t^{(xy)}t^yt\\
&y(x^5y)t^{(x^5y)}t^{(xy)}t^yt\\
&(1,9,4,12,7,2,10,5,13,8,3,11,6)t_7t_{11}t_{12}t_1.
\end{aligned}$$

Likewise, $tx^4ytx^4ytx^3y = t_9t_2t_9$. Thus,

$$M = \langle N, t_7t_{11}t_{12}t_1, t_9t_2t_9\rangle = 2^{\bullet}(13:2)\ where|M| = 104.$$

Then M is the subgroup.
Now that we have obtain the homomorphic image and maximal subgroup we can start constructing the a manual double coset enumeration of G over the maximal subgroup, M and N. Let w be a word in t_is and $[w]$ be the double coset, MwN.

MeN

The process on constructing a manual double coset, we begin with the first double coset $MeN = \{Ne^n|n\epsilon N\}$ denotes by $[*]$. In this double coset there is only one single coset, namely M. The coset stabiliser of M is N and is transitive on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$. Thus, has a single orbit {1,2,3,4,5,6,7,8,9,10,11,12,13}. Take a representative from the single orbit and do right multiplication to Me. Implies $Met_1 = Mt_1N$ is a new double coset denoted as [1]. Since we have 13 elements on the orbit, that means 13 things will go to the new double coset, [1].

Mt_1N

Continuing with the double coset Mt_1N, we find the point stabiliser N^1 and coset stabiliser $N^{(1)}$ to determine the amount of single cosets that are in the new double coset [1]

$$N^{(1)} \geq \langle (2,13)(3,12)(4,11)(5,10)(6,9)(7,8) \rangle.$$

Thus, the order of the coset stabiliser of $N^{(1)}$ denoted as $|N^{(1)}| = 2$. So the number of single cosets in $N^{(1)}$ is $\frac{|N|}{|N^{(1)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(1)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{1\}, \{2,13\}, \{3,12\}, \{4,11\}, \{5,10\}, \{6,9\}\{7,8\}$$

Take a representative t_i from each orbit to determine if any double cosets Mt_1t_i are new.

$$\begin{aligned}
& Mt_1t_1 = Me, \in [*](1 \textit{ loop back}) \\
& Mt_1t_2 \in [12], \\
& Mt_1t_3 \in [13], \\
& Mt_1t_4 \in [14], \\
& \quad t_1t_5 = x^4yx^5yt_7t_{11}t_{12}t_1t_8t_6 \\
\implies & Mt_1t_5 = Mt_8t_6 \in [13] \\
& \qquad (\textit{since } \{N(t_1t_3)^n | n \in N\} \textit{ and } x^4yx^5yt_7t_{11}t_{12}t_1 \in M), \\
& Mt_1t_6 \in [16], \\
& \quad t_1t_7 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_1 \\
\implies & Mt_1t_7 = Mt_1 \in [1](2 \textit{ loop back}) \\
& \qquad (\textit{since } \{N(t_1)^n | n \in N\} \textit{ and } yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M)
\end{aligned}$$

After multiplying on the right by an element from each orbit, we get new double cosets with the following double cosets $Mt_1t_2N, Mt_1t_3N, Mt_1t_4N$, and Mt_1t_6N single coset representatives, $Mt_1t_2, Mt_1t_3, Mt_1t_4$, and Mt_1t_6. We labeled each single coset representative as [12], [13], [14], and [16], respectively. And we checked and proved for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_2N}$

From the previous work we were able to determine our new double cosets with the single coset representatives [12], [13], [14], and [16]. Continuing with our work we start with the first double coset Mt_1t_2N and find the coset stabilizer. But Mt_1t_2 is not distinct, since $t_1t_2 = xt_9t_8$ where $Mt_9t_8 \ \epsilon \ Mt_1t_2$. Now $M(t_1t_2)^{(1,9)(2,8)(3,7)(4,6)(10,13)(11,12)}$ $= Mt_9t_8$. Hence, $(1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \ \epsilon \ N^{(12)}$. Therefore:

$$N^{(12)} \geq \langle(1,9)(2,8)(3,7)(4,6)(10,13)(11,12)\rangle$$

Thus the order of $N^{(12)} = 2$, therefore, the number of single cosets in Mt_1t_2N is $\frac{N}{N^{(12)}}=\frac{26}{2} = 13$. Now we find the orbits of $N^{(12)}$ on thirteen letters $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8,$ $t_9, t_{10}, t_{11}, t_{12}, t_{13}$ which are:

$$\mathcal{O} = \{5\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{10,13\}, \{11,12\}$$

Now we take a representative from each orbit and we do right multiply to the single coset Mt_1t_2 and check if we get new double cosets.

$$\begin{aligned}
&Mt_1t_2t_5 \in [125] \\
&Mt_1t_2t_1 = Id(G)t_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_{13}t_8 \\
\implies &Mt_1t_2t_1 = Mt_{13}t_8 \in [16] \\
&\qquad (since\ \{N(t_1t_6)^n | n \in N\}\ and\ Id(G)t_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M), \\
&Mt_1t_2t_2 = Mt_1 \in [1] \\
&Mt_1t_2t_3 \in [123] \\
&Mt_1t_2t_4 \in [124] \\
&Mt_1t_2t_{10} \in [1210] \\
&Mt_1t_2t_{11} \in [1211]
\end{aligned}$$

After right multiply by an element from each orbit, the new double cosets $Mt_1t_2t_5N$, $Mt_1t_2t_3N$, $Mt_1t_2t_4N$, $Mt_1t_2t_{10}N$, $Mt_1t_2t_{11}N$ with single coset representatives $Mt_1t_2t_5$, $Mt_1t_2t_3$, $Mt_1t_2t_4$, $Mt_1t_2t_{10}$, $Mt_1t_2t_{11}$, denoted as [125], [123], [124], [1210], and [1211].

$\boldsymbol{Mt_1t_2t_5N}$

Following the same process from above, the double coset $Mt_1t_2t_5N$ has coset stabilizer of $N^{(125)} = N^{125}$. But $Mt_1t_2t_5$ is not distinct, since $t_1t_2t_5 = xt_9t_8t_5$ where $Mt_9t_8t_5 \epsilon Mt_1t_2t_5$. Now $M(t_1t_2t_5)^{(1,9)(2,8)(3,7)(4,6)(10,13)(11,12)} = Mt_9t_8t_5$. Hence, $(1,9)$ $(2,8)(3,7)(4,6)(10,13)(11,12) \epsilon N^{(125)}$. Therefore:

$$N^{(125)} \geq \langle (1,9)(2,8)(3,7)(4,6)(10,13)(11,12) \rangle$$

The order of $N^{(125)} = 2$. Thus the number of single cosets in $Mt_1t_2t_5N$ is $\frac{N}{N^{(125)}} = \frac{26}{2} = 13$. Now that we know 13 single cosets exist in [125] so we find the orbits of $N^{(125)}$ on thirteen letters. The orbits of

$$\mathcal{O} = \{5\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{10,13\}, \{11,12\}$$

Take a representative from each orbit and do right multiply of the single coset $Mt_1t_2t_5$.

$$\begin{aligned}
& Mt_1t_2t_5t_5 = Mt_1t_2 \in [12] \\
& Mt_1t_2t_5t_1 = x^4yx^4yx^3t_4t_{11}t_4t_1t_2t_5 \\
\implies & Mt_1t_2t_5t_1 = Mt_1t_2t_5 \in [125] \\
& \qquad (since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M), \\
& Mt_1t_2t_5t_2 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_8t_9t_{12} \\
\implies & Mt_1t_2t_5t_2 = Mt_8t_9t_{12} \in [125] \\
& \qquad (since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M), \\
& Mt_1t_2t_5t_3 = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_4t_7t_4 \\
\implies & Mt_1t_2t_5t_3 = Mt_4t_7t_4 \in [141] \\
& \qquad (since\ \{N(t_1t_4t_1)^n | n \in N\}\ and\ xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M), \\
& Mt_1t_2t_5t_4 = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_8t_6t_1 \\
\implies & Mt_1t_2t_5t_4 = Mt_8t_6t_1 \in [138] \\
& \qquad (since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),
\end{aligned}$$

$$Mt_1t_2t_5t_{10} = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_3t_6t_9$$
$$\implies Mt_1t_2t_5t_{10} = Mt_3t_6t_9 \in [147]$$
$$(since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_2t_5t_{11} = xyx^4yx^4y^2t_{12}t_5t_{12}t_2t_1t_5$$
$$\implies Mt_1t_2t_5t_{11} = Mt_2t_1t_5 \in [1211]$$
$$(since\ \{N(t_1t_2t_{11})^n | n \in N\}\ and\ xyx^4yx^4y^2t_{12}t_5t_{12} \in M)$$

After checking if there are any new cosets, the results listen above tells us there are no new double cosets. Therefore, we must check and prove which single cosets are equal to other existing double cosets.

$\boldsymbol{Mt_1t_2t_3N}$

Continuing with the new double coset $Mt_1t_2t_3N$, we find the point stabiliser N^{123} and coset stabiliser $N^{(123)}$ to determine the amount of single cosets that are in the new double coset [123].

$$N^{(123)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(123)}$ denoted as $|N^{(123)}| = 1$. So the number of single cosets in $N^{(123)}$ is $\frac{|N|}{|N^{(123)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(123)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_2t_3t_i$ exist.

$$Mt_1t_2t_3t_1 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_1t_2t_3$$
$$\implies Mt_1t_2t_3t_1 = Mt_1t_2t_3 \in [123]$$
$$(since\ \{N(t_1t_2t_3)^n | n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_2t_3t_2 = x^4yx^4yxt_1t_2t_9t_2t_{10}t_8t_{10}$$
$$\implies Mt_1t_2t_3t_2 = Mt_{10}t_8t_{10} \in [131]$$

$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ x^4yx^4yxt_1t_2t_9t_2 \in M),$

$Mt_1t_2t_3t_3 = Mt_1t_2 \in [12]$

$Mt_1t_2t_3t_4 = x^4yx^4yx^3t_4t_{11}t_4t_9t_{11}t_3$

$\implies Mt_1t_2t_3t_4 = Mt_9t_{11}t_3 \in [138]$

$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M)$

$Mt_1t_2t_3t_5 = x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11}t_8t_{10}t_8$

$\implies Mt_1t_2t_3t_5 = Mt_8t_{10}t_8 \in [131]$

$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11} \in M),$

$Mt_1t_2t_3t_6 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_{12}t_{13}t_2$

$\implies Mt_1t_2t_3t_6 = Mt_{12}t_{13}t_2 \in [124]$

$(since\ \{N(t_1t_2t_4)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$

$Mt_1t_2t_3t_7 = x^9t_{10}t_6t_5t_3t_8t_{10}t_{13}t_4$

$\implies Mt_1t_2t_3t_7 = Mt_8t_{10}t_{13}t_4 \in [13610]$

$(since\ \{N(t_1t_3t_6t_{10})^n|n \in N\}\ and\ x^9t_{10}t_6t_5t_3 \in M),$

$Mt_1t_2t_3t_8 = x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11}t_5t_8t_{10}$

$\implies Mt_1t_2t_3t_8 = Mt_5t_8t_{10} \in [146]$

$(since\ \{N(t_1t_4t_6)^n|n \in N\}\ and\ x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11} \in M),$

$Mt_1t_2t_3t_9 = xt_2t_{10}$

$\implies Mt_1t_2t_3t_9 = Mt_2t_{10} \in [16]$

$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ x \in M),$

$Mt_1t_2t_3t_{10} = x^4yx^4yx^3t_4t_{11}t_4t_6t_7t_2$

$\implies Mt_1t_2t_3t_{10} = Mt_6t_7t_2 \in [1210]$

$(since\ \{N(t_1t_2t_{10})^n|n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M),$

$Mt_1t_2t_3t_{11} = x^4yx^4yt_1t_8t_1t_2t_7t_{12}$

$\implies Mt_1t_2t_3t_{11} = Mt_2t_7t_{12} \in [1611]$

$(since\ \{N(t_1t_6t_{11})^n|n \in N\}\ and\ x^4yx^4yt_1t_8t_1 \in M),$

$$Mt_1t_2t_3t_{12} = x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1t_2t_5t_8$$
$$\implies Mt_1t_2t_3t_{12} = Mt_2t_5t_8 \in [147]$$
$$(since\ \{N(t_1t_4t_7)^n|n \in N\}\ and\ x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1 \in M),$$
$$Mt_1t_2t_3t_{13} = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_1t_4t_5$$
$$\implies Mt_1t_2t_3t_{13} = Mt_1t_4t_5 \in [145]$$
$$(since\ \{N(t_1t_4t_5)^n|n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_2t_4N}$

Moving on with the new double coset $Mt_1t_2t_4N$, we find the point stabiliser N^{124} and coset stabiliser $N^{(124)}$ to determine the amount of single cosets that are in the new double coset [124]. But $Mt_1t_2t_4$ is not distinct, since $t_1t_2t_4 = t_6t_5t_3$ where $Mt_6t_5t_3$ ϵ $Mt_1t_2t_4$. Now $M(t_1t_2t_4)^{(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)} = Mt_6t_5t_3$.
Hence, $(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)\ \epsilon\ N^{(124)}$.

$$N^{(124)} \geq \rangle(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)\langle.$$

Hence, the order of the coset stabiliser of $N^{(124)}$ denoted as $|N^{(124)}| = 2$. So the number of single cosets in $N^{(124)}$ is $\frac{|N|}{|N^{(124)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(124)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{10\},\{1,6\},\{2,5\},\{3,4\},\{7,13\},\{8,12\},\{9,11\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_2t_4t_i$ exist.

$$Mt_1t_2t_4t_{10} = x^4yx^4yx^4t_5t_{12}t_5t_{13}t_7t_8t_3$$
$$\implies Mt_1t_2t_4t_{10} = Mt_7t_8t_3 \in [1210]$$
$$(since\ \{N(t_1t_2t_{10})^n|n \in N\}\ and\ x^4yx^4yx^4t_5t_{12}t_5t_{13} \in M),$$

$$Mt_1t_2t_4t_1 = xyx^{-2}y^2x^{-1}yt_5t_2t_1t_{13}t_2t_{13}t_{12}t_5t_2t_{12}$$

$$\implies Mt_1t_2t_4t_1 = Mt_5t_2t_{12} \in [147]$$

$$(since\ \{N(t_1t_4t_7)^n|n \in N\}\ and\ xyx^{-2}y^2x^{-1}yt_5t_2t_1t_{13}t_2t_{13}t_{12} \in M),$$

$$Mt_1t_2t_4t_2 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_7t_4t_3$$

$$\implies Mt_1t_2t_4t_2 = Mt_7t_4t_3 \in [145]$$

$$(since\ \{N(t_1t_4t_5)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M)$$

$$Mt_1t_2t_4t_4 = Mt_1t_2 \in [12]$$

$$Mt_1t_2t_4t_7 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_{11}t_9t_{12}$$

$$\implies Mt_1t_2t_4t_7 = Mt_{11}t_9t_{12} \in [1313]$$

$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_2t_4t_8 = x^5y^2x^3t_8t_2t_4t_3t_1t_3t_4t_5$$

$$\implies Mt_1t_2t_4t_8 = Mt_3t_4t_5 \in [123]$$

$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ x^5y^2x^3t_8t_2t_4t_3t_1 \in M),$$

$$Mt_1t_2t_4t_9 = x^4yx^4yx^3t_4t_{11}t_4t_6t_7t_3$$

$$\implies Mt_1t_2t_4t_9 = Mt_6t_7t_3 \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_2t_{10}N}$

Following the same process from above, the double coset $Mt_1t_2t_{10}N$ has coset stabilizer of $N^{(1210)} = N^{1210} = \langle(1,7)(2,6)(3,5)(8,13)(9,12)(10,11)\rangle$. Since $t_1t_2t_{10} = xt_7t_6t_{11}$ where $Mt_7t_6t_{11}\ \epsilon\ Mt_1t_2t_{10}$. Now $M(t_1t_2t_{10})^{(1,7)(2,6)(3,5)(8,13)(9,12)(10,11)}$ $= Mt_7t_6t_{11}$. Hence, $(1,7)(2,6)(3,5)(8,13)(9,12)(10,11)\ \epsilon\ N^{(1210)}$. The order of $N^{(1210)}$ $= 2$. Thus the number of single cosets in $Mt_1t_2t_{10}N$ is $\frac{N}{N^{(1210)}}=\frac{26}{2} = 13$. Now that we know 13 single cosets exist in [1210] lets find the orbits of $N^{(1210)}$ on thirteen letters. The orbits are

$$\mathcal{O} = \{4\}, \{1,7\}, \{2,6\}, \{3,5\}, \{8,13\}, \{9,12\}, \{10,11\}$$

Take a representative from each orbit and do right multiply to the single coset $Mt_1t_2t_{10}t_i$.

$$Mt_1t_2t_{10}t_4 = x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1t_8t_9t_{11}$$

$$\implies Mt_1t_2t_{10}t_4 = Mt_9t_8t_{11} \in [124]$$

$$(since\ \{N(t_1t_2t_4)^n | n \in N\}\ and\ x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1 \in M),$$

$$Mt_1t_2t_{10}t_1 = yx^{-1}y^2xyt_5t_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_5t_{13}t_8$$

$$\implies Mt_1t_2t_{10}t_1 = Mt_5t_{13}t_8 \in [1611]$$

$$(since\ \{N(t_1t_6t_{11})^n | n \in N\}\ and\ yx^{-1}y^2xyt_5t_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_2t_{10}t_2 = t_3t_1$$

$$\implies Mt_1t_2t_{10}t_2 = Mt_3t_1 \in [13]$$

$$(since\ \{N(t_1t_3)^n | n \in N\}\ and\ t_3t_1 \in M)$$

$$Mt_1t_2t_{10}t_3 = x^4yx^4yx^4t_5t_{12}t_5t_{12}t_{11}t_{10}$$

$$\implies Mt_1t_2t_{10}t_3 = Mt_{12}t_{11}t_{10} \in [123]$$

$$(since\ \{N(t_1t_2t_3)^n | n \in N\}\ and\ x^4yx^4yx^4t_5t_{12}t_5 \in M),$$

$$Mt_1t_2t_{10}t_8 = xt_4t_9$$

$$\implies Mt_1t_2t_{10}t_8 = Mt_4t_9 \in [16]$$

$$(since\ \{N(t_1t_6)^n | n \in N\}\ and\ x \in M),$$

$$Mt_1t_2t_{10}t_9 = x^4yx^4yx^2t_3t_{10}t_3t_8t_{10}t_2$$

$$\implies Mt_1t_2t_{10}t_9 = Mt_8t_{10}t_2 \in [138]$$

$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^4yx^4yx^2t_3t_{10}t_3 \in M)$$

$$Mt_1t_2t_{10}t_{10} = Mt_1t_2 \in [12]$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we check and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_2t_{11}N}$

Continuing with the new double coset $Mt_1t_2t_{11}N$, we find the point stabiliser

N^{1211} and coset stabiliser $N^{(1211)}$ to determine the amount of single cosets that are in the new double coset [1211].

$$N^{(1211)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(1211)}$ denoted as $|N^{(1211)}| = 1$. So the number of single cosets in $N^{(1211)}$ is $\frac{|N|}{|N^{(1211)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(1211)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_2t_{11}t_i$ exist.

$$\begin{aligned}
& Mt_1t_2t_{11}t_1 = xyx^4yx^4y^2t_{12}t_5t_{12}t_9t_6t_3 \\
\implies & Mt_1t_2t_{11}t_1 = Mt_9t_6t_3 \in [147] \\
& \qquad (since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ xyx^4yx^4y^2t_{12}t_5t_{12} \in M), \\
& Mt_1t_2t_{11}t_2 = x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1t_{13}t_5 \\
\implies & Mt_1t_2t_{11}t_2 = Mt_{13}t_5 \in [16] \\
& \qquad (since\ \{N(t_1t_6)^n | n \in N\}\ and\ x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1 \in M), \\
& Mt_1t_2t_{11}t_3 = x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1t_1t_{11} \\
\implies & Mt_1t_2t_{11}t_3 = Mt_1t_{11} \in [14] \\
& \qquad (since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1 \in M), \\
& Mt_1t_2t_{11}t_4 = x^2yx^4yx^4y^2t_{12}t_5t_{12}t_9t_{10}t_{12} \\
\implies & Mt_1t_2t_{11}t_4 = Mt_9t_{10}t_{12} \in [124] \\
& \qquad (since\ \{N(t_1t_2t_4)^n | n \in N\}\ and\ x^2yx^4yx^4y^2t_{12}t_5t_{12} \in M), \\
& Mt_1t_2t_{11}t_5 = xyx^4yx^4y^2t_{12}t_5t_{12}t_2t_1t_{11} \\
\implies & Mt_1t_2t_{11}t_5 = Mt_2t_1t_{11} \in [125] \\
& \qquad (since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ xyx^4yx^4y^2t_{12}t_5t_{12} \in M),
\end{aligned}$$

$$Mt_1t_2t_{11}t_6 = xt_4t_7$$

$$\implies Mt_1t_2t_{11}t_6 = Mt_4t_7 \in [14]$$

$$(since\ \{N(t_1t_4)^n | n \in N\}\ and\ x \in M),$$

$$Mt_1t_2t_{11}t_7 = x^{-1}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_4t_2t_{12}$$

$$\implies Mt_1t_2t_{11}t_7 = Mt_4t_2t_{12} \in [136]$$

$$(since\ \{N(t_1t_3t_6)^n | n \in N\}\ and\ x^{-1}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12} \in M),$$

$$Mt_1t_2t_{11}t_8 = x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1t_3t_5t_{10}$$

$$\implies Mt_1t_2t_{11}t_8 = Mt_3t_5t_{10} \in [138]$$

$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^6y^2x^2t_9t_8t_3t_1t_3t_2t_1 \in M),$$

$$Mt_1t_2t_{11}t_9 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_{11}t_1t_3$$

$$\implies Mt_1t_2t_{11}t_9 = Mt_1t_1t_3 \in [146]$$

$$(since\ \{N(t_1t_4t_6)^n | n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$

$$Mt_1t_2t_{11}t_{10} = x^4y^2x^3t_8t_2t_4t_3t_1t_{12}t_2t_3$$

$$\implies Mt_1t_2t_{11}t_{10} = Mt_{12}t_2t_3 \in [145]$$

$$(since\ \{N(t_1t_4t_5)^n | n \in N\}\ and\ x^4y^2x^3t_8t_2t_4t_3t_1 \in M),$$

$$Mt_1t_2t_{11}t_{11} = Mt_1t_2 \in [12]$$

$$Mt_1t_2t_{11}t_{12} = x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1t_9t_7t_2$$

$$\implies Mt_1t_2t_{11}t_{12} = Mt_9t_7t_2 \in [138]$$

$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$

$$Mt_1t_2t_{11}t_{13} = yx^5yt_7t_{11}t_{12}t_1t_9t_{11}$$

$$\implies Mt_1t_2t_{11}t_{13} = Mt_9t_{11} \in [13]$$

$$(since\ \{N(t_1t_3)^n | n \in N\}\ and\ yx^5yt_7t_{11}t_{12}t_1 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_3N}$

Following the same process from above, the double coset Mt_1t_3N has coset stabilizer of $N^{(13)} = N^{13} = \langle e \rangle$. The order of $N^{(13)} = 1$. Thus the number of single cosets in Mt_1t_3N is $\frac{N}{N^{(13)}} = \frac{26}{1} = 26$. Now that we know 26 single cosets exist in [13] lets find the orbits of $N^{(13)}$ on thirteen letters. The orbits are

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative from each orbit and do right multiply to the single coset $Mt_1t_3t_i$

$$Mt_1t_3t_1 \in [131],$$

$$Mt_1t_3t_2 = x^{-1}t_{10}t_{11}t_6$$

$$\implies Mt_1t_3t_2 = Mt_{10}t_{11}t_6 \in [1210]$$

$$(since\ \{N(t_1t_2t_{10})^n | n \in N\}\ and\ x^{-1} \in M),$$

$$Mt_1t_3t_3 = Mt_1 \in [1],$$

$$Mt_1t_3t_4 = x^3yx^5yt_7t_{11}t_{12}t_1t_8$$

$$\implies Mt_1t_3t_4 = Mt_8 \in [1]$$

$$(since\ \{N(t_1)^n | n \in N\}\ and\ x^3yx^5yt_7t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_3t_5 = x^3yx^5yt_7t_{11}t_{12}t_1t_6t_7t_3$$

$$\implies Mt_1t_3t_5 = Mt_6t_7t_3 \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n | n \in N\}\ and\ x^3yx^5yt_7t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_3t_6 \in [136],$$

$$Mt_1t_3t_7 \in [137],$$

$$Mt_1t_3t_8 \in [138],$$

$$Mt_1t_3t_9 = x^4yx^4yt_4t_{11}t_4t_{11}t_3$$

$$\implies Mt_1t_3t_9 = Mt_{11}t_3 \in [16]$$

$$(since\ \{N(t_1t_6)^n | n \in N\}\ and\ x^4yx^4yt_4t_{11}t_4 \in M),$$

$$Mt_1t_3t_{10} = x^4y^2xy^2t_6t_{13}t_2t_1t_{12}t_6t_4$$

$$\implies Mt_1t_3t_{10} = Mt_6t_4 \in [13]$$

$$(since\ \{N(t_1t_3)^n | n \in N\}\ and\ x^4y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$

$$Mt_1t_3t_{11} = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_7t_4$$
$$\implies Mt_1t_3t_{11} = Mt_7t_4 \in [14]$$
$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$
$$Mt_1t_3t_{12} = x^9t_{10}t_6t_5t_3t_{13}t_{11}t_8$$
$$\implies Mt_1t_3t_{12} = Mt_{13}t_{11}t_8 \in [136]$$
$$(since\ \{N(t_1t_3t_6)^n|n \in N\}\ and\ x^9t_{10}t_6t_5t_3 \in M), Mt_1t_3t_{13} \in [1313]$$

After right multiply by an element from each orbit, the new double cosets $Mt_1t_3t_1N$, $Mt_1t_3t_6N$, $Mt_1t_3t_7N$, $Mt_1t_3t_8N$, $Mt_1t_3t_{13}N$ with single coset representatives are $Mt_1t_3t_1$, $Mt_1t_3t_6$, $Mt_1t_3t_7$, $Mt_1t_3t_8$, $Mt_1t_3t_{13}$, denoted as [131], [136], [137], [138], and [1313].

$\boldsymbol{Mt_1t_3t_1N}$

Using the process from above, new double coset $Mt_1t_3t_1N$, we find the point stabiliser N^{131} and coset stabiliser $N^{(131)}$ to determine the amount of single cosets that are in the new double coset [131].

$$N^{(131)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(131)}$ denoted as $|N^{(131)}| = 1$. So the number of single cosets in $N^{(131)}$ is $\frac{|N|}{|N^{(131)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(131)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_1t_i$ exist.

$$Mt_1t_3t_1t_1 = Mt_1t_3 \in [13]$$
$$Mt_1t_3t_1t_2 = et_2t_{10}$$
$$\implies Mt_1t_3t_1t_2 = Mt_2t_{10} \in [16]$$
$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ e \in M),$$

$$Mt_1t_3t_1t_3 = x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11}t_6t_9t_{11}$$
$$\implies Mt_1t_3t_1t_3 = Mt_6t_9t_{11} \in [146]$$
$$(since\ \{N(t_1t_4t_6)^n|n \in N\}\ and\ x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11} \in M),$$
$$Mt_1t_3t_1t_4 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_{11}t_3t_8$$
$$\implies Mt_1t_3t_1t_4 = Mt_{11}t_3t_8 \in [1611]$$
$$(since\ \{N(t_1t_6t_{11})^n|n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_3t_1t_5 = yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1t_8t_5t_2$$
$$\implies Mt_1t_3t_1t_5 = Mt_8t_5t_2 \in [147]$$
$$(since\ \{N(t_1t_4t_7)^n|n \in N\}\ and\ yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$
$$Mt_1t_3t_1t_6 = x^5t_8t_{10}t_{13}t_4$$
$$\implies Mt_1t_3t_1t_6 = Mt_8t_{10}t_{13}t_4 \in [13610]$$
$$(since\ \{N(t_1t_3t_6t_{10})^n|n \in N\}\ and\ x^5 \in M),$$
$$Mt_1t_3t_1t_7 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_7t_5t_1$$
$$\implies Mt_1t_3t_1t_7 = Mt_7t_5t_1 \in [137]$$
$$(since\ \{N(t_1t_3t_7)^n|n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_3t_1t_8 = x^4yx^4yt_1t_8t_1t_8t_6t_1$$
$$\implies Mt_1t_3t_1t_8 = Mt_8t_6t_1 \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^4yx^4yt_1t_8t_1 \in M),$$
$$Mt_1t_3t_1t_9 = x^4yx^4yx^3t_4t_{11}t_4t_{10}t_9t_8$$
$$\implies Mt_1t_3t_1t_9 = Mt_{10}t_9t_8 \in [123]$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M),$$
$$Mt_1t_3t_1t_{10} = x^2yx^5yt_7t_{11}t_{12}t_1t_7t_{12}$$
$$\implies Mt_1t_3t_1t_{10} = Mt_7t_{12} \in [16]$$
$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ x^2yx^5yt_7t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_3t_1t_{11} = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_7t_8t_9$$
$$\implies Mt_1t_3t_1t_{11} = Mt_7t_8t_9 \in [123]$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_3t_1t_{12} = x^7t_8t_4t_3t_1t_2t_4t_7t_{11}$$
$$\implies Mt_1t_3t_1t_{12} = Mt_1t_2t_4t_7t_{11} \in [13610]$$
$$(since\ \{N(t_1t_3t_6t_{10})^n|n \in N\}\ and\ x^7t_8t_4t_3 \in M),$$
$$Mt_1t_3t_1t_{13} = x^5y^2x^3t_8t_2t_4t_3t_1t_{13}t_3t_4$$
$$\implies Mt_1t_3t_1t_{13} = Mt_{13}t_3t_4 \in [145]$$
$$(since\ \{N(t_1t_4t_5)^n|n \in N\}\ and\ x^5y^2x^3t_8t_2t_4t_3t_1 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_3t_6N}$

Moving on with the new double coset $Mt_1t_3t_6N$, we find the point stabiliser N^{136} and coset stabiliser $N^{(136)}$ to determine the amount of single cosets that are in the new double coset [136]. But $Mt_1t_3t_6$ is not distinct, since $t_1t_3t_6 = x^{-6}t_6t_4t_1$ where $Mt_6t_4t_1 \ \epsilon \ Mt_1t_3t_6$. Now $M(t_1t_3t_6)^{(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)} = Mt_6t_4t_1$. Hence, $(1,6)(2,5)(3,4)(7,13)(8,12)(9,11) \ \epsilon \ N^{(136)}$.

$$N^{(136)} \geq \langle (1,6)(2,5)(3,4)(7,13)(8,12)(9,11) \rangle.$$

Hence, the order of the coset stabiliser of $N^{(136)}$ denoted as $|N^{(136)}| = 2$. So the number of single cosets in $N^{(136)}$ is $\frac{|N|}{|N^{(136)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(136)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{10\}\{1,6\},\{2,5\},\{3,4\},\{7,13\},\{8,12\},\{9,11\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_6t_i$ exist.

$$Mt_1t_3t_6t_{10} \in [13610]$$
$$Mt_1t_3t_6t_1 = x^{-6}t_6t_4$$
$$\implies Mt_1t_3t_6t_1 = Mt_6t_4 \in [13]$$

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^{-6} \in M),$

$$Mt_1t_3t_6t_2 = x^2yx^5yt_7t_{11}t_{12}t_1t_{13}t_{11}$$

$$\implies Mt_1t_3t_6t_2 = Mt_{13}t_{11} \in [13]$$

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^2yx^5yt_7t_{11}t_{12}t_1 \in M),$

$$Mt_1t_3t_6t_3 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_1t_4t_7$$

$$\implies Mt_1t_3t_6t_3 = Mt_1t_4t_7 \in [147]$$

$(since\ \{N(t_1t_4t_7)^n|n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$

$$Mt_1t_3t_6t_7 = x^4yx^4yx^2t_3t_{10}t_3t_2t_{13}t_3$$

$$\implies Mt_1t_3t_6t_7 = Mt_2t_{13}t_3 \in [1313]$$

$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^4yx^4yx^2t_3t_{10}t_3 \in M),$

$$Mt_1t_3t_6t_8 = x^4y^2x^3t_8t_2t_4t_3t_1t_2t_{13}t_{10}$$

$$\implies Mt_1t_3t_6t_8 = Mt_2t_{13}t_{10} \in [136]$$

$(since\ \{N(t_1t_3t_6)^n|n \in N\}\ and\ x^4y^2x^3t_8t_2t_4t_3t_1 \in M),$

$$Mt_1t_3t_6t_9 = xyx^4yx^4y^2t_{12}t_5t_{12}t_3t_4t_{13}$$

$$\implies Mt_1t_3t_6t_9 = Mt_3t_4t_{13} \in [1211]$$

$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ xyx^4yx^4y^2t_{12}t_5t_{12} \in M)$

After checking if there are any new cosets, the results listen above tells us there are no new double cosets. Therefore, we must have checked and prove which single cosets are equal to other existing double cosets.

$Mt_1t_3t_7N$

For the new double coset $Mt_1t_3t_7N$ we find the point stabiliser N^{137} and coset stabiliser $N^{(137)}$ to determine the amount of single cosets that are in the new double coset [137]. But $Mt_1t_3t_7$ is not distinct, since $t_1t_3t_7 = x^5t_2t_{13}t_9$ where $Mt_2t_{13}t_9$ ϵ $Mt_1t_3t_7$. Now $M(t_1t_3t_7)^{(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)} = Mt_2t_{13}t_9$. Hence, $(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)\ \epsilon\ N^{(137)}$.

$$N^{(137)} \geq \langle(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)\rangle.$$

Hence, the order of the coset stabiliser of $N^{(137)}$ denoted as $|N^{(137)}| = 2$. So the number of single cosets in $N^{(137)}$ is $\frac{|N|}{|N^{(137)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(137)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{8\}\{1, 2\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{7, 9\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_7t_i$ exist.

$$\begin{aligned}
&Mt_1t_3t_7t_8 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_1t_4t_5 \\
\implies &Mt_1t_3t_7t_8 = Mt_1t_4t_5 \in [145] \\
&\qquad (since\ \{N(t_1t_4t_5)^n | n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M), \\
&Mt_1t_3t_7t_1 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_7t_5t_7 \\
\implies &Mt_1t_3t_7t_1 = Mt_7t_5t_7 \in [131] \\
&\qquad (since\ \{N(t_1t_3t_1)^n | n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M), \\
&Mt_1t_3t_7t_3 = x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_{10}t_{12}t_9 \\
\implies &Mt_1t_3t_7t_3 = Mt_{10}t_{12}t_9 \in [1313] \\
&\qquad (since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M), \\
&Mt_1t_3t_7t_4 = x^3yx^5yt_7t_{11}t_{12}t_1t_5t_8 \\
\implies &Mt_1t_3t_7t_4 = Mt_5t_8 \in [14] \\
&\qquad (since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^3yx^5yt_7t_{11}t_{12}t_1 \in M), \\
&Mt_1t_3t_7t_5 = yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1t_6t_9t_{12} \\
\implies &Mt_1t_3t_7t_5 = Mt_6t_9t_{12} \in [147] \\
&\qquad (since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M), \\
&Mt_1t_3t_7t_6 = x^5y^2x^3t_8t_2t_4t_3t_1t_6t_3t_1 \\
\implies &Mt_1t_3t_7t_6 = Mt_6t_3t_1 \in [146] \\
&\qquad (since\ \{N(t_1t_4t_6)^n | n \in N\}\ and\ x^5y^2x^3t_8t_2t_4t_3t_1 \in M) \\
&Mt_1t_3t_7t_7 = Mt_1t_3 \in [13]
\end{aligned}$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets if they are equal to other existing double cosets.

$\boldsymbol{Mt_1t_3t_8N}$

Continuing with the new double coset $Mt_1t_3t_8N$, we find the point stabiliser N^{138} and coset stabiliser $N^{(138)}$ to determine the amount of single cosets that are in the new double coset [138].

$$N^{(138)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(138)}$ denoted as $|N^{(138)}| = 1$. So the number of single cosets in $N^{(138)}$ is $\frac{|N|}{|N^{(138)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(138)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_8t_i$ exist.

$$
\begin{aligned}
&Mt_1t_3t_8t_1 = x^4yx^4yt_1t_8t_1t_8t_6t_8\\
\Longrightarrow\ &Mt_1t_3t_8t_1 = Mt_8t_6t_8 \in [131]\\
&\qquad\qquad (since\ \{N(t_1t_3t_1)^n | n \in N\}\ and\ x^4yx^4yt_1t_8t_1 \in M),\\
&Mt_1t_3t_8t_2 = x^4yx^4yxt_2t_9t_2t_7t_8t_3\\
\Longrightarrow\ &Mt_1t_3t_8t_2 = Mt_7t_8t_3 \in [1210]\\
&\qquad\qquad (since\ \{N(t_1t_2t_{10})^n | n \in N\}\ and\ x^4yx^4yxt_2t_9t_2 \in M),\\
&Mt_1t_3t_8t_3 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_9t_1t_6\\
\Longrightarrow\ &Mt_1t_3t_8t_3 = Mt_9t_1t_6 \in [1611]\\
&\qquad\qquad (since\ \{N(t_1t_6t_{11})^n | n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),\\
&Mt_1t_3t_8t_4 = x^{10}t_{11}t_7t_6t_4t_9t_{12}\\
\Longrightarrow\ &Mt_1t_3t_8t_4 = Mt_9t_{12} \in [14]\\
&\qquad\qquad (since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^{10}t_{11}t_7t_6t_4 \in M),
\end{aligned}
$$

$$Mt_1t_3t_8t_5 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_{13}t_1t_4$$

$$\implies Mt_1t_3t_8t_5 = Mt_{13}t_1t_4 \in [125]$$

$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_3t_8t_6 = x^4yx^4yxt_2t_9t_2t_{12}t_{13}t_9$$

$$\implies Mt_1t_3t_8t_6 = Mt_{12}t_{13}t_9 \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^4yx^4yxt_2t_9t_2 \in M),$$

$$Mt_1t_3t_8t_7 = x^{-1}t_1t_9$$

$$\implies Mt_1t_3t_8t_7 = Mt_1t_9 \in [16]$$

$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ x^{-1} \in M),$$

$$Mt_1t_3t_8t_8 = Mt_1t_3 \in [13]$$

$$Mt_1t_3t_8t_9 = x^4yx^4yt_1t_8t_1t_6t_7t_8$$

$$\implies Mt_1t_3t_8t_9 = Mt_6t_7t_8 \in [123]$$

$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ x^4yx^4yt_1t_8t_1 \in M),$$

$$Mt_1t_3t_8t_{10} = x^4yx^4yx^2t_3t_{10}t_3t_7t_{10}t_7$$

$$\implies Mt_1t_3t_8t_{10} = Mt_7t_{10}t_7 \in [141]$$

$$(since\ \{N(t_1t_4t_1)^n|n \in N\}\ and\ x^4yx^4yx^2t_3t_{10}t_3 \in M),$$

$$Mt_1t_3t_8t_{11} = x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1t_9t_8t_{12}$$

$$\implies Mt_1t_3t_8t_{11} = Mt_9t_8t_{12} \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$

$$Mt_1t_3t_8t_{12} = yxx^4yx^4y^2t_{12}t_5t_{12}t_{11}t_1t_4$$

$$\implies Mt_1t_3t_8t_{12} = Mt_{11}t_1t_4 \in [147]$$

$$(since\ \{N(t_1t_4t_7)^n|n \in N\}\ and\ yxx^4yx^4y^2t_{12}t_5t_{12} \in M),$$

$$Mt_1t_3t_8t_{13} = yx^5yt_7t_9t_{10}t_1t_8t_3$$

$$\implies Mt_1t_3t_8t_{13} = Mt_8t_3 \in [16]$$

$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ yx^5yt_7t_9t_{10}t_1 \in M)$$

After checking if there are any new cosets, the results listen above tells us there are no new double cosets. Therefore, we must have checked and prove which single cosets are equal to other existing double cosets.

$Mt_1t_3t_{13}N$

For the new double coset $Mt_1t_3t_{13}N$ we find the point stabiliser N^{1313} and coset stabiliser $N^{(1313)}$ to determine the amount of single cosets that are in the new double coset [1313]. But $Mt_1t_3t_{13}$ is not distinct, since $t_1t_3t_{13} = x^{-3}t_1t_{12}t_2$ where $Mt_1t_{12}t_2 \in Mt_1t_3t_{13}$. Now $M(t_1t_3t_{13})^{(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)} = Mt_1t_{12}t_2$. Hence, $(2,13)(3,12)(4,11)(5,10)(6,9)(7,8) \in N^{(1313)}$.

$$N^{(1313)} \geq \langle (2,13)(3,12)(4,11)(5,10)(6,9)(7,8) \rangle.$$

Hence, the order of the coset stabiliser of $N^{(1313)}$ denoted as $|N^{(1313)}| = 2$. So the number of single cosets in $N^{(1313)}$ is $\frac{|N|}{|N^{(1313)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(1313)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{1\}\{2,13\},\{3,12\},\{4,11\},\{5,10\},\{6,9\},\{7,8\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_{13}t_i$ exist.

$$Mt_1t_3t_{13}t_1 = yx^4y^2xyt_{13}t_{11}t_{12}t_1t_7t_4t_2$$
$$\implies Mt_1t_3t_{13}t_1 = Mt_7t_4t_2 \in [146]$$
$$(since\ \{N(t_1t_4t_6)^n | n \in N\}\ and\ yx^4y^2xyt_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_3t_{13}t_{13} = Mt_1t_3 \in [13],$$
$$Mt_1t_3t_{13}t_3 = x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_1t_{12}t_2$$
$$\implies Mt_1t_3t_{13}t_3 = Mt_1t_{12}t_2 \in [1313]$$
$$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M),$$
$$Mt_1t_3t_{13}t_4 = x^6y^2x^2t_9t_8t_3t_1t_1t_2t_1t_6t_9t_{12}$$
$$\implies Mt_1t_3t_{13}t_4 = Mt_6t_9t_{12} \in [147]$$
$$(since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ x^6y^2x^2t_9t_8t_3t_1t_1t_2t_1 \in M),$$

$$Mt_1t_3t_{13}t_5 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_6t_7t_9$$

$$\implies Mt_1t_3t_{13}t_5 = Mt_6t_7t_9 \in [124]$$

$$(since\ \{N(t_1t_2t_4)^n|n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_3t_{13}t_6 = x^{-1}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_5t_3t_{13}$$

$$\implies Mt_1t_3t_{13}t_6 = Mt_5t_3t_{13} \in [136]$$

$$(since\ \{N(t_1t_3t_6)^n|n \in N\}\ and\ x^{-1}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M),$$

$$Mt_1t_3t_{13}t_7 = x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_6t_4t_{13}$$

$$\implies Mt_1t_3t_{13}t_7 = Mt_6t_4t_{13} \in [137]$$

$$(since\ \{N(t_1t_3t_7)^n|n \in N\}\ and\ x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we check and prove for those double cosets if they are equal to other existing double cosets.

$\boldsymbol{Mt_1t_4N}$

Continuing with the new double coset Mt_1t_4N, we find the point stabiliser N^{14} and coset stabiliser $N^{(14)}$ to determine the amount of single cosets that are in the new double coset [14].

$$N^{(14)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(14)}$ denoted as $|N^{(14)}| = 1$. So the number of single cosets in $N^{(14)}$ is $\frac{|N|}{|N^{(14)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(14)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_i$ exist.

$$Mt_1t_4t_1 \in [141]$$

$$Mt_1t_4t_2 = x^{10}t_{11}t_7t_6t_4t_8t_{11}t_8$$

$$\implies Mt_1t_4t_2 = Mt_8t_{11}t_8 \in [141]$$

$$(since\ \{N(t_1t_4t_1)^n|n \in N\}\ and\ x^{10}t_{11}t_7t_6t_4 \in M),$$

$$Mt_1t_4t_3 = x^{-1}t_{11}t_{12}t_8$$

$$\implies Mt_1t_4t_3 = Mt_{11}t_{12}t_8 \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^{-1} \in M),$$

$$Mt_1t_4t_4 = Mt_1 \in [1]$$

$$Mt_1t_4t_5 \in [145]$$

$$Mt_1t_4t_6 \in [146]$$

$$Mt_1t_4t_7 \in [147]$$

$$Mt_1t_4t_8 = x^4y^2xy^2t_6t_{13}t_2t_1t_{12}t_2t_{12}$$

$$\implies Mt_1t_4t_8 = Mt_2t_{12} \in [14]$$

$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^4y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$

$$Mt_1t_4t_9 = x^7t_8t_4t_3t_1t_6t_8t_{13}$$

$$\implies Mt_1t_3t_9 = Mt_6t_8t_{13} \in [138]$$

$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^7t_8t_4t_3t_1 \in M),$$

$$Mt_1t_4t_{10} = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_7t_5$$

$$\implies Mt_1t_4t_{10} = Mt_7t_5 \in [13]$$

$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$

$$Mt_1t_4t_{11} = x^2yx^4yx^4y^2t_{12}t_5t_{12}t_1t_4$$

$$\implies Mt_1t_4t_{11} = Mt_1t_4 \in [14]$$

$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^2yx^4yx^4y^2t_{12}t_5t_{12} \in M),$$

$$Mt_1t_4t_{12} = x^2yx^5yt_7t_{11}t_{12}t_1t_{13}t_4$$

$$\implies Mt_1t_4t_{12} = Mt_1t_{13}t_4 \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^2yx^5yt_7t_{11}t_{12} \in M),$$

$$Mt_1t_4t_{13} = x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1t_{11}t_9t_5$$

$$\implies Mt_1t_4t_{13} = Mt_{11}t_9t_5 \in [137]$$

$$(since\ \{N(t_1t_3t_7)^n|n \in N\}\ and\ x^2yx^3yxt_1t_{12}t_{10}t_{12}t_2t_1 \in M)$$

After right multiplying by an element from each orbit, the new double cosets $Mt_1t_4t_1N$, $Mt_1t_4t_5N$, $Mt_1t_4t_6N$, $Mt_1t_4t_7N$ with single coset representatives are $Mt_1t_4t_1$, $Mt_1t_4t_5$,

$Mt_1t_4t_6$, $Mt_1t_4t_7$, denoted as [141], [145], [146], and [147].

$\boldsymbol{Mt_1t_4t_1N}$

For the new double coset $Mt_1t_4t_1N$ we find the point stabiliser N^{141} and coset stabiliser $N^{(141)}$ to determine the amount of single cosets that are in the new double coset [141]. But $Mt_1t_4t_1$ is not distinct, since $t_1t_4t_1 = t_5t_2t_5$ where $Mt_5t_2t_5 \ \epsilon \ Mt_1t_4t_1$. Now $M(t_1t_4t_1)^{(1,5)(2,4)(6,13)(7,12)(8,11)(9,10)} = Mt_5t_2t_5$.
Hence, $(1,5)(2,4)(6,13)(7,12)(8,11)(9,10) \ \epsilon \ N^{(141)}$.

$$N^{(141)} \geq \langle (1,5)(2,4)(6,13)(7,12)(8,11)(9,10) \rangle.$$

Hence, the order of the coset stabiliser of $N^{(141)}$ denoted as $|N^{(141)}| = 2$. So the number of single cosets in $N^{(141)}$ is $\frac{|N|}{|N^{(141)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(141)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{3\}, \{1,5\}, \{2,4\}, \{6,13\}, \{7,12\}, \{8,11\}, \{9,10\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_1t_i$ exist.

$$Mt_1t_4t_1t_1 = Mt_1t_4 \in [14]$$
$$Mt_1t_4t_1t_2 = x^4yx^4yx^4t_5t_{12}t_5t_{11}t_9t_4$$
$$\implies Mt_1t_4t_1t_2 = Mt_{11}t_9t_4 \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^4yx^4yx^4t_5t_{12}t_5 \in M),$$
$$Mt_1t_4t_1t_3 = x^4y^2x^3t_8t_2t_4t_3t_1t_1t_4t_1$$
$$\implies Mt_1t_4t_1t_3 = Mt_1t_4t_1 \in [141]$$
$$(since\ \{N(t_1t_4t_1)^n | n \in N\}\ and\ x^4y^2x^3t_8t_2t_4t_3 \in M),$$
$$Mt_1t_4t_1t_8 = x^4yx^4yx^3t_4t_{11}t_4t_8t_{10}t_2$$
$$\implies Mt_1t_4t_1t_8 = Mt_8t_{10}t_2 \in [1611]$$
$$(since\ \{N(t_1t_6t_11)^n | n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M),$$

$$Mt_1t_4t_1t_9 = et_5t_2$$
$$\implies Mt_1t_4t_1t_9 = Mt_5t_2 \in [14]$$
$$(since\ \{N(t_1t_4)^n | n \in N\}\ and\ e \in M),$$
$$Mt_1t_4t_1t_6 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_{13}t_1t_4$$
$$\implies Mt_1t_4t_1t_6 = Mt_{13}t_1t_4 \in [125]$$
$$(since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_4t_1t_7 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_7t_4t_1$$
$$\implies Mt_1t_4t_1t_7 = Mt_7t_4t_1 \in [147]$$
$$(since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_4t_5N}$

Moving on with the new double coset $Mt_1t_4t_5N$, we find the point stabiliser N^{145} and coset stabiliser $N^{(145)}$ to determine the amount of single cosets that are in the new double coset [145]. But $Mt_1t_4t_5$ is not distinct, since $t_1t_4t_5 = xt_2t_{12}t_{11}$ where $Mt_2t_{12}t_{11}\ \epsilon\ Mt_1t_4t_5$. Now $M(t_1t_4t_5)^{(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)} = Mt_2t_{12}t_{11}$. Hence, $(1,2)(3,13)(4,12)(5,11)(6,10)(7,9)\ \epsilon\ N^{(145)}$.

$$N^{(145)} \geq \langle (1,2)(3,13)(4,12)(5,11)(6,10)(7,9) \rangle.$$

Hence, the order of the coset stabiliser of $N^{(145)}$ denoted as $|N^{(145)}| = 2$. So the number of single cosets in $N^{(145)}$ is $\frac{|N|}{|N^{(145)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(145)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{8\}\{1,2\},\{3,13\},\{4,12\},\{5,11\},\{6,10\},\{7,9\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_5t_i$ exist.

$$Mt_1t_4t_5t_8 = x^4y^2xy^2t_6t_{13}t_2t_1t_{12}t_1t_3t_7$$
$$\implies Mt_1t_4t_5t_8 = Mt_1t_3t_7 \in [137]$$
$$(since\ \{N(t_1t_3t_7)^n|n \in N\}\ and\ x^4y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$
$$Mt_1t_4t_5t_1 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_2t_4t_2$$
$$\implies Mt_1t_4t_5t_1 = Mt_2t_4t_2 \in [131]$$
$$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$
$$Mt_1t_4t_5t_3 = x^4yx^4yxt_2t_9t_2t_2t_1t_{13}$$
$$\implies Mt_1t_4t_5t_3 = Mt_2t_1t_{13} \in [123]$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ x^4yx^4yxt_2t_9t_2 \in M),$$
$$Mt_1t_4t_5t_4 = x^5y^2x^3t_8t_2t_4t_3t_1t_{13}t_{12}t_3$$
$$\implies Mt_1t_4t_5t_4 = Mt_{13}t_{12}t_3 \in [1211]$$
$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^5y^2x^3t_8t_2t_4t_3t_1 \in M),$$
$$Mt_1t_4t_5t_5 = Mt_1t_4 \in [14]$$
$$Mt_1t_4t_5t_6 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_2t_3t_5$$
$$\implies Mt_1t_4t_5t_6 = Mt_2t_3t_5 \in [124]$$
$$(since\ \{N(t_1t_2t_4)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$
$$Mt_1t_4t_5t_7 = x^8t_9t_5t_4t_2t_7t_9t_{12}t_3$$
$$\implies Mt_1t_4t_5t_7 = Mt_7t_9t_{12}t_3 \in [13610]$$
$$(since\ \{N(t_1t_2t_4)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M)$$

After checking if there are any new cosets, the results listen above tells us there are no new double cosets. Therefore, we must check and prove which single cosets are equal to other existing double cosets.

$\boldsymbol{Mt_1t_4t_6N}$

For the new double coset $Mt_1t_4t_6N$ we find the point stabiliser N^{146} and coset stabiliser $N^{(146)}$ to determine the amount of single cosets that are in the new double coset [146]. But $Mt_1t_4t_6$ is not distinct, since $t_1t_4t_6 = x^{-6}t_{13}t_{10}t_8$ where $Mt_{13}t_{10}t_8$ ϵ

$Mt_1t_4t_6$. Now $M(t_1t_4t_6)^{(1,13)(2,12)(3,11)(4,10)(5,9)(6,8)} = Mt_{13}t_{10}t_8$.
Hence, $(1,13)(2,12)(3,11)(4,10)(5,9)(6,8) \epsilon N^{(146)}$.

$$N^{(146)} \geq \langle(1,13)(2,12)(3,11)(4,10)(5,9)(6,8)\rangle.$$

Hence, the order of the coset stabiliser of $N^{(146)}$ denoted as $|N^{(146)}| = 2$. So the number of single cosets in $N^{(146)}$ is $\frac{|N|}{|N^{(146)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(146)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{7\}\{1,13\},\{2,12\},\{3,11\},\{4,10\},\{5,9\},\{6,8\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_6t_i$ exist.

$$Mt_1t_4t_6t_7 = yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_7t_5t_6$$
$$\implies Mt_1t_4t_6t_7 = Mt_7t_5t_6 \in [1313]$$
$$(\textit{since } \{N(t_1t_3t_{13})^n | n \in N\} \textit{ and } yx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_4t_6t_1 = x^5y^2x^3t_{10}t_9t_{10}t_{12}t_1t_2t_1t_{10}t_9t_{13}$$
$$\implies Mt_1t_4t_6t_1 = Mt_{10}t_9t_{13} \in [137]$$
$$(\textit{since } \{N(t_1t_3t_7)^n | n \in N\} \textit{ and } x^5y^2x^3t_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$
$$Mt_1t_4t_6t_2 = yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1t_{10}t_9t_{13}$$
$$\implies Mt_1t_4t_6t_2 = Mt_{10}t_9t_{13} \in [1211]$$
$$(\textit{since } \{N(t_1t_2t_{11})^n | n \in N\} \textit{ and } yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$
$$Mt_1t_4t_6t_3 = x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1t_5t_3t_5$$
$$\implies Mt_1t_4t_6t_3 = Mt_5t_3t_5 \in [131]$$
$$(\textit{since } \{N(t_1t_3t_1)^n | n \in N\} \textit{ and } x^{-4}t_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$
$$Mt_1t_4t_6t_4 = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_{10}t_{11}t_{12}$$
$$\implies Mt_1t_4t_6t_4 = Mt_{10}t_{11}t_{12} \in [123]$$
$$(\textit{since } \{N(t_1t_2t_3)^n | n \in N\} \textit{ and } xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_4t_6t_5 = x^{-1}t_1t_9$$
$$\implies Mt_1t_4t_6t_5 = Mt_1t_9 \in [16]$$
$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ x^{-1} \in M)$$
$$Mt_1t_4t_6t_6 = Mt_1t_4 \in [14]$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_4t_7N}$

Continuing with the new double coset $Mt_1t_4t_7N$, we find the point stabiliser N^{147} and coset stabiliser $N^{(147)}$ to determine the amount of single cosets that are in the new double coset [147].

$$N^{(147)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(147)}$ denoted as $|N^{(147)}| = 1$. So the number of single cosets in $N^{(147)}$ is $\frac{|N|}{|N^{(147)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(147)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_7t_i$ exist.

$$Mt_1t_4t_7t_1 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_3t_6t_3$$
$$\implies Mt_1t_4t_7t_1 = Mt_3t_6t_3 \in [141]$$
$$(since\ \{N(t_1t_4t_1)^n|n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$
$$Mt_1t_4t_7t_2 = x^4yx^4t_1t_8t_1t_4t_6t_{11}$$
$$\implies Mt_1t_4t_7t_2 = Mt_4t_6t_{11} \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^4yx^4t_1t_8t_1 \in M),$$
$$Mt_1t_4t_7t_3 = yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1t_1t_3t_6$$

$$\Longrightarrow Mt_1t_4t_7t_3 = Mt_1t_3t_6 \in [136]$$

$$(since\ \{N(t_1t_3t_6)^n | n \in N\}\ and\ yx^{-1}y^2xyt_1t_{12}t_{11}t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_4t_7t_4 = yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1t_8t_6t_8$$

$$\Longrightarrow Mt_1t_4t_7t_4 = Mt_8t_6t_8 \in [131]$$

$$(since\ \{N(t_1t_3t_1)^n | n \in N\}\ and\ yx^2yt_{10}t_9t_{10}t_{12}t_1t_2t_1 \in M),$$

$$Mt_1t_4t_7t_5 = xyx^{-2}y^2x^{-1}yt_5t_2t_1t_{13}t_2t_{13}t_{12}t_{13}t_1t_3$$

$$\Longrightarrow Mt_1t_4t_7t_5 = Mt_{13}t_1t_3 \in [124]$$

$$(since\ \{N(t_1t_2t_4)^n | n \in N\}\ and\ xyx^{-2}y^2x^{-1}yt_5t_2t_1t_{13}t_2t_{13}t_{12} \in M),$$

$$Mt_1t_4t_7t_6 = x^4yx^4yx^4t_5t_{12}t_5t_1t_4t_7$$

$$\Longrightarrow Mt_1t_4t_7t_6 = Mt_1t_4t_7 \in [147]$$

$$(since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ x^4yx^4yx^4t_5t_{12}t_5 \in M),$$

$$Mt_1t_4t_7t_7 = Mt_1t_4 \in [14],$$

$$Mt_1t_4t_7t_8 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_7t_6t_3$$

$$\Longrightarrow Mt_1t_4t_7t_8 = Mt_7t_6t_3 \in [125]$$

$$(since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$

$$Mt_1t_4t_7t_9 = yx^4yx^4y^2t_{12}t_5t_{12}t_9t_8t_{12}$$

$$\Longrightarrow Mt_1t_4t_7t_9 = Mt_9t_8t_{12} \in [1211]$$

$$(since\ \{N(t_1t_2t_{11})^n | n \in N\}\ and\ yx^4yx^4y^2t_{12}t_5t_{12} \in M),$$

$$Mt_1t_4t_7t_{10} = x^5t_{12}t_{10}t_7t_3$$

$$\Longrightarrow Mt_1t_4t_7t_{10} = Mt_{12}t_{10}t_7t_3 \in [13610]$$

$$(since\ \{N(t_1t_3t_6t_{10})^n | n \in N\}\ and\ x^5 \in M),$$

$$Mt_1t_4t_7t_{11} = x^4yx^4yx^3t_4t_{11}t_4t_{13}t_1t_2$$

$$\Longrightarrow Mt_1t_4t_7t_{11} = Mt_{13}t_1t_2 \in [123]$$

$$(since\ \{N(t_1t_2t_3)^n | n \in N\}\ and\ x^4yx^4yx^3t_4t_{11}t_4 \in M),$$

$$Mt_1t_4t_7t_{12} = yx^4yx^4y^2t_{12}t_5t_{12}t_9t_{11}t_8$$

$$\Longrightarrow Mt_1t_4t_7t_{12} = Mt_9t_{11}t_8 \in [1313]$$

$$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ yx^4yx^4y^2t_{12}t_5t_{12} \in M),$$

$$Mt_1t_4t_7t_{13} = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_{10}t_8t_4$$
$$\implies Mt_1t_4t_7t_{13} = Mt_{10}t_8t_4 \in [137]$$
$$(since\ \{N(t_1t_3t_7)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M)$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus we have checked and prove for those double cosets that are equal to other existing double cosets.

$\boldsymbol{Mt_1t_6N}$

Using the process from above, new double coset Mt_1t_6N, we find the point stabiliser N^{16} and coset stabiliser $N^{(16)}$ to determine the amount of single cosets that are in the new double coset [16].

$$N^{(16)} \geq \langle e \rangle.$$

Thus, the order of the coset stabiliser of $N^{(16)}$ denoted as $|N^{(16)}| = 1$. So the number of single cosets in $N^{(16)}$ is $\frac{|N|}{|N^{(16)}|} = \frac{26}{1} = 26$. Now we find the orbits of $N^{(16)}$ on the transversals $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\{12\}, \{13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_6t_i$ exist.

$$Mt_1t_6t_1 = t_2t_{13}t_2$$
$$\implies Mt_1t_6t_1 = Mt_2t_{13}t_2 \in [131]$$
$$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ e \in M),$$
$$Mt_1t_6t_2 = x^5y^2xy^2t_6t_{13}t_2t_1t_{12}t_3t_{11}$$
$$\implies Mt_1t_6t_2 = Mt_3t_{11} \in [16]$$
$$(since\ \{N(t_1t_6)^n|n \in N\}\ and\ x^5y^2xy^2t_6t_{13}t_2t_1t_{12} \in M),$$
$$Mt_1t_6t_3 = x^4yx^5yt_7t_{11}t_{12}t_1t_2t_3t_{12}$$
$$\implies Mt_1t_6t_3 = Mt_2t_3t_{12} \in [1211]$$
$$(since\ \{N(t_1t_2t_{11})^n|n \in N\}\ and\ x^4yx^5yt_7t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_6t_4 = xyx^5yt_7t_{11}t_{12}t_8t_{10}t_8$$
$$\implies Mt_1t_6t_4 = Mt_8t_{10}t_8 \in [131]$$
$$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ xyx^5yt_7t_{11}t_{12} \in M),$$
$$Mt_1t_6t_5 = x^{-1}t_{11}t_{12}t_7$$
$$\implies Mt_1t_6t_5 = Mt_{11}t_{12}t_7 \in [1210]$$
$$(since\ \{N(t_1t_2t_{10})^n|n \in N\}\ and\ x^{-1} \in M),$$
$$Mt_1t_6t_6 = Mt_1 \in [1]$$
$$Mt_1t_6t_7 = xt_2t_1t_{13}$$
$$\implies Mt_1t_6t_7 = Mt_2t_1t_{13} \in [123]$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ x \in M),$$
$$Mt_1t_6t_8 = x^{-1}t_1t_{12}t_7$$
$$\implies Mt_1t_6t_8 = Mt_1t_{12}t_7 \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^{-1} \in M),$$
$$Mt_1t_6t_9 = x^9t_{10}t_6t_5t_3t_8t_6t_1$$
$$\implies Mt_1t_6t_9 = Mt_8t_6t_1 \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^9t_{10}t_6t_5t_3 \in M),$$
$$Mt_1t_6t_{10} = x^5t_2t_5t_7$$
$$\implies Mt_1t_6t_{10} = Mt_2t_5t_7 \in [146]$$
$$(since\ \{N(t_1t_4t_6)^n|n \in N\}\ and\ x^5 \in M),$$
$$Mt_1t_6t_{11} \in [1611],$$
$$Mt_1t_6t_{12} = x^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_4t_6$$
$$\implies Mt_1t_6t_{12} = Mt_4t_6 \in [13]$$
$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M),$$
$$Mt_1t_6t_{13} = x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1t_{13}t_{12}$$
$$\implies Mt_1t_6t_{13} = Mt_{13}t_{12} \in [12]$$
$$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^{-2}y^2t_{12}t_{11}t_{12}t_{13}t_1t_{12}t_1 \in M)$$

After right multiply by an element from each orbit, the new double coset $Mt_1t_6t_{11}N$

with single coset representatives is $Mt_1t_6t_{11}$, denoted as [1611].

$\boldsymbol{Mt_1t_6t_{11}N}$

For the new double coset $Mt_1t_6t_{11}N$ we find the point stabiliser N^{1611} and coset stabiliser $N^{(1611)}$ to determine the amount of single cosets that are in the new double coset [1611]. But $Mt_1t_6t_{11}$ is not distinct, since $t_1t_6t_{11} = x^9t_{10}t_6t_5t_3t_1t_{10}t_5t_{13}$ where $Mt_{10}t_5t_{13} \ \epsilon \ Mt_1t_6t_{11}$. Now $M(t_1t_6t_{11})^{(1,10)(2,9)(3,8)(4,7)(5,6)(11,13)} = Mt_{10}t_5t_{13}$. Hence, $(1,10)(2,9)(3,8)(4,7)(5,6)(11,13) \ \epsilon \ N^{(1611)}$.

$$N^{(1411)} \geq \langle (1,10)(2,9)(3,8)(4,7)(5,6)(11,13) \rangle.$$

Hence, the order of the coset stabiliser of $N^{(146)}$ denoted as $|N^{(146)}| = 2$. So the number of single cosets in $N^{(146)}$ is $\frac{|N|}{|N^{(146)}|} = \frac{26}{2} = 13$. Now we find the orbits of $N^{(146)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{12\}\{1,10\},\{2,9\},\{3,8\},\{4,7\},\{5,6\},\{11,13\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_4t_{11}t_i$ exist.

$$Mt_1t_6t_{11}t_{12} = xyx^4yx^4y^2t_{12}t_5t_{12}t_1t_6t_{11}$$

$$\Longrightarrow Mt_1t_6t_{11}t_{12} = Mt_1t_6t_{11} \in [1611]$$

$$(since\ \{N(t_1t_6t_{11})^n | n \in N\}\ and\ xyx^4yx^4y^2t_{12}t_5t_{12} \in M),$$

$$Mt_1t_6t_{11}t_1 = x^{-4}t_{10}t_9t_{10}t_{13}t_1t_2t_1t_{11}t_{10}t_9$$

$$\Longrightarrow Mt_1t_6t_{11}t_1 = Mt_{11}t_{10}t_9 \in [123]$$

$$(since\ \{N(t_1t_2t_3)^n | n \in N\}\ and\ x^{-4}t_{10}t_9t_{10}t_{13}t_1t_2t_1 \in M),$$

$$Mt_1t_6t_{11}t_2 = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_7t_{10}t_7$$

$$\Longrightarrow Mt_1t_6t_{11}t_2 = Mt_7t_{10}t_7 \in [141]$$

$$(since\ \{N(t_1t_4t_1)^n | n \in N\}\ and\ xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M),$$

$$Mt_1t_6t_{11}t_3 = x^4yx^4yxt_2t_9t_2t_5t_3t_{11}$$
$$\implies Mt_1t_6t_{11}t_3 = Mt_5t_3t_{11} \in [138]$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^4yx^4yxt_2t_9t_2 \in M),$$
$$Mt_1t_6t_{11}t_4 = x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11}t_7t_5t_7$$
$$\implies Mt_1t_6t_{11}t_4 = Mt_7t_5t_7 \in [131]$$
$$(since\ \{N(t_1t_3t_1)^n|n \in N\}\ and\ x^4y^2xyxyt_5t_{12}t_1t_{13}t_{11} \in M),$$
$$Mt_1t_6t_{11}t_5 = yx^4yx^4y^2t_{12}t_5t_{12}t_5t_4t_9$$
$$\implies Mt_1t_6t_{11}t_5 = Mt_5t_4t_9 \in [1210]$$
$$(since\ \{N(t_1t_2t_{10})^n|n \in N\}\ and\ yx^4yx^4y^2t_{12}t_5t_{12} \in M),$$
$$Mt_1t_6t_{11}t_{11} = Mt_1t_6 \in [16]$$

After checking if there are any new cosets, the results listen above tells us there are no new double cosets. Therefore, we must check and prove which single cosets are equal to other existing double cosets.

$\boldsymbol{Mt_1t_3t_6t_{10}N}$

Using the process from above, new double coset $Mt_1t_3t_6t_{10}N$, we find the point stabiliser N^{13610} and coset stabiliser $N^{(13610)}$ to determine the amount of single cosets that are in the new double coset [13610]. But $Mt_1t_3t_6t_{10}$ is not distinct, since $t_1t_3t_6t_{10} = x^{-6}t_6t_4t_1t_{10}$ where $Mt_6t_4t_1t_{10}\ \epsilon\ Mt_1t_3t_6t_{10}$. Now $M(t_1t_3t_6t_{10})^{(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)}$ $= Mt_6t_4t_1t_{10}$. Hence, $(1,6)(2,5)(3,4)(7,13)(8,12)(9,11)\ \epsilon\ N^{(13610)}$.

$$N^{(13610)} \geq \langle (1,6)(2,5)(3,4)(7,13)(8,12)(9,11) \rangle.$$

Thus, the order of the coset stabiliser of $N^{(13610)}$ denoted as $|N^{(13610)}| = 2$. So the number of single cosets in $N^{(13610)}$ is $\frac{|N|}{|N^{(13610)}|} = \frac{26}{1} = 13$. Now we find the orbits of $N^{(13610)}$ on the transversals $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ which are:

$$\mathcal{O} = \{10\}, \{1,6\}, \{2,5\}, \{3,4\}, \{7,13\}, \{8,12\}, \{9,11\}$$

Take a representative t_i from each orbit to determine if any double cosets $Mt_1t_3t_6t_{10}t_i$ exist.

$$
\begin{aligned}
& Mt_1t_3t_6t_{10}t_{10} = Mt_1t_3t_6 \in [136], \\
& Mt_1t_3t_6t_{10}t_1 = x^{10}t_{11}t_7t_6t_4t_9t_6t_5 \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_1 = Mt_9t_6t_5 \in [145] \\
& \qquad (since\ \{N(t_1t_4t_5)^n | n \in N\}\ and\ x^{10}t_{11}t_7t_6t_4 \in M), \\
& Mt_1t_3t_6t_{10}t_2 = xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1t_{11}t_{13}t_3t_7 \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_2 = Mt_{11}t_{13}t_3t_7 \in [13610] \\
& \qquad (since\ \{N(t_1t_3t_6t_{10})^n | n \in N\}\ and\ xyx^4y^2xyt_7t_{13}t_{11}t_{12}t_1 \in M), \\
& Mt_1t_3t_6t_{10}t_3 = x^5t_{12}t_9t_6 \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_3 = Mt_{12}t_9t_6 \in [147] \\
& \qquad (since\ \{N(t_1t_4t_7)^n | n \in N\}\ and\ x^5 \in M), \\
& Mt_1t_3t_6t_{10}t_7 = x^8t_9t_5t_4t_2t_{13}t_{12}t_{11} \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_7 = Mt_{13}t_{12}t_{11} \in [123] \\
& \qquad (since\ \{N(t_1t_2t_3)^n | n \in N\}\ and\ x^8t_9t_5t_4t_2 \in M), \\
& Mt_1t_3t_6t_{10}t_8 = x^{-1}t_{13}t_{11}t_{13} \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_8 = Mt_{13}t_{11}t_{13} \in [131] \\
& \qquad (since\ \{N(t_1t_3t_1)^n | n \in N\}\ and\ x^{-1} \in M), \\
& Mt_1t_3t_6t_{10}t_{11} = yx^5yt_7t_9t_{10}t_1t_{13}t_2t_{13} \\
\Longrightarrow\ & Mt_1t_3t_6t_{10}t_{11} = Mt_{13}t_2t_{13} \in [131] \\
& \qquad (since\ \{N(t_1t_3t_1)^n | n \in N\}\ and\ yx^5yt_7t_9t_{10}t_1 \in M)
\end{aligned}
$$

After multiplying on the right by an element from each orbit, there are no new double cosets. Thus, we have checked and prove for those double cosets that are equal to other existing double cosets.

We have completed the double coset enumeration of G, since the set of right cosets is closed under right multiplication. Thus the index of N in G is 378. We have concluded the following:

$$G = MeN \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_3N \cup Mt_1t_4N \cup Mt_1t_6N \cup Mt_1t_2t_5N$$
$$\cup Mt_1t_2t_3N \cup Mt_1t_2t_4N \cup Mt_1t_2t_{10}N \cup Mt_1t_2t_{11}N \cup Mt_1t_3t_1N$$
$$\cup Mt_1t_3t_6N \cup Mt_1t_3t_7N \cup Mt_1t_3t_8N \cup Mt_1t_3t_{13}N \cup Mt_1t_4t_1N$$
$$\cup Mt_1t_4t_5N \cup Mt_1t_4t_6N \cup Mt_1t_4t_7N \cup Mt_1t_6t_{11}N \cup Mt_1t_3t_6t_{10}N$$

where

$$G = \frac{2^{*13} : (13 : 2)}{((x^4)tt^x)^3, ((x^6)tt^x)^2} \cong 2 \times PGL_2(27).$$

Therefore,

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(124)}|} + \frac{|N|}{|N^{(1210)}|} + \frac{|N|}{|N^{(1211)}|} + \frac{|N|}{|N^{(13)}|}$$

$$+ \frac{|N|}{|N^{(131)}|} + \frac{|N|}{|N^{(136)}|} + \frac{|N|}{|N^{(137)}|} + \frac{|N|}{|N^{(138)}|} + \frac{|N|}{|N^{(1313)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(141)}|}$$

$$+ \frac{|N|}{|N^{(145)}|} + \frac{|N|}{|N^{(146)}|} + \frac{|N|}{|N^{(147)}|} + \frac{|N|}{|N^{(16)}|} + \frac{|N|}{|N^{(1611)}|} + \frac{|N|}{|N^{(13610)}|} \times |M|$$

and

$|G| \leq (1 + 13 + 13 + 13 + 26 + 13 + 13 + 26 + 26 + 26 + 13 + 13 + 26 + 13 + 26 + 13 + 13 + 13 + 26 + 26 + 13 + 13) \times 104$

$|G| \leq 39312.$

The Cayley graph summarizes the information listed above.

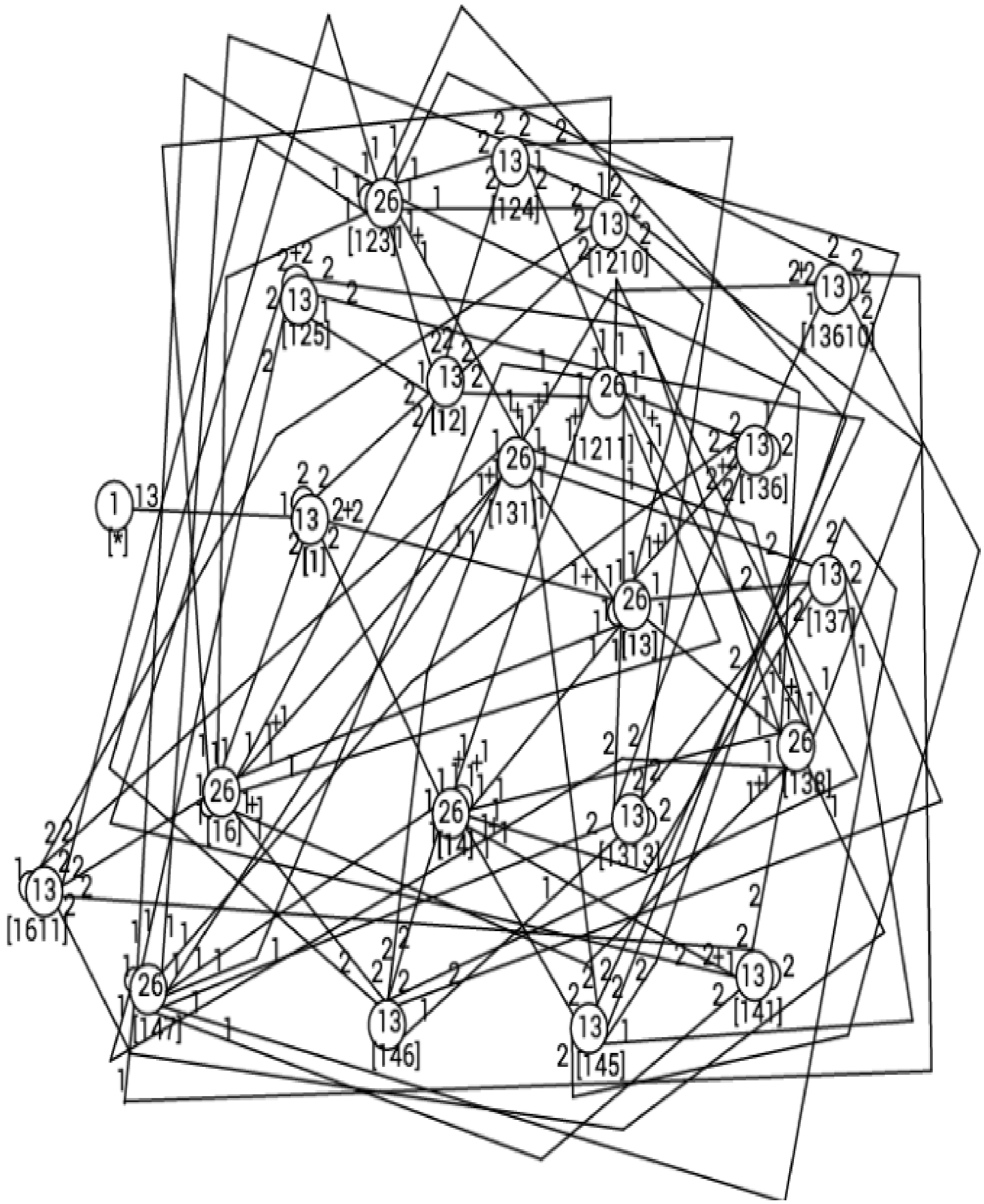

Figure 6.1: Cayley graph of $2 \times PGL_2(27)$ over $M = 2^{\bullet} : (13 : 2)$

6.2 Construct of $PSL_3(3)$ over $M = (13:3)$

6.2.1 Double Coset Enumeration of G

Before we apply the double coset enumeration over a maximal subgroup, we are going to factor the progenitor $2^{*13} : 13$ by a special relations, $(x^{-4}t)^3$, $(x^2t)^4$, and $(x^{-1}t)^4$, denoted by:

$$\langle x, t | x^{-13},$$

$$t^2,$$

$$(x^{-4}t)^3, (x^{-2}t)^4, (x^{-1}t)^4\rangle$$

and

$$N \cong 13 =$$

$$\langle x | x^{-13}\rangle.$$

So we obtained the homomorphic image:

$$G \cong \frac{2^{*13} : 13}{(x^{-4}t)^3, (x^2t)^4, (x^{-1}t)^4} \cong PSL_3(3),$$

where $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$ and $t \sim t_1$.

Let $\pi = x^{-4} = (1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5)$, then the relation $(x^{-4}t_1)^3 = 1$ can be written as

$1 = (\pi t_1)^3$, which it can be expand:

$$\begin{aligned} (\pi t_1)^3 &= 1 \\ \pi^3 t_1^{\pi^2} t_1^{\pi} t_1 &= 1 \\ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) t_6 t_{10} t_1 &= 1 \\ t_1 t_{10} &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) t_6 \end{aligned}$$

The second relations is written as follow:

Let $\beta = x^{-2}$

$$\begin{aligned}
(\beta t_1)^4 &= 1\\
\beta^4 t_1^{\beta^3} t_1^{\beta^2} t_1^{\beta} t_1 &= 1\\
(1,6,11,3,8,13,5,10,2,7,12,4,9)t_8 t_{10} t_{12} t_1 &= 1\\
t_1 t_{12} = (1,6,11,3,8,13,5,10,2,7,12,4,9)t_8 t_{10}
\end{aligned}$$

And the third relation can be expand as follow:
Let $\alpha = x^{-1}$

$$\begin{aligned}
(\alpha t_1)^4 &= 1\\
\alpha^4 t_1^{\alpha^3} t_1^{\alpha^2} t_1^{\alpha} t_1 &= 1\\
(1,10,6,2,11,7,3,12,8,4,13,9,5)t_{11} t_{12} t_{13} t_1 &= 1\\
t_1 t_{13} = (1,10,6,2,11,7,3,12,8,4,13,9,5)t_{11} t_{12}
\end{aligned}$$

Let M be the maximal subgroup generated by the control group $N = 13$ and $t_8 t_{10} t_9 = t^{x^8} t^{x^{10}} t^{x^9}$

$$M = \langle N, t^{x^8} t^{x^{10}} t^{x^9} \rangle, = (13:3) \text{ where } |M| = 39.$$

Then M is the maximal subgroup.

Let us start constructing a manual double coset enumeration of G over the maximal subgroup, M and N. Denote $[w]$ to be the double coset MwN, where w is a word in $t_i's$.

MeN

We begin with the double coset MeN, denote $[*]$. This double coset contains only one single coset, namely M. The single coset stabilizer of M is N, which is transitive on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ and therefore, has a single orbit,

$$\mathcal{O} = \{\{1,2,3,4,5,6,7,8,9,10,11,12,13\}\}.$$

Take an element from $\mathcal{O}$ say t_1 and multiply the single coset representative M by t_1

to obtain Mt_1. This is a new double coset Mt_1N, denote it [1] so thirteen symmetric generators will go to our new double coset, [1].

Mt_1N

Continuing with the double coset Mt_1N, we find the point stabilizer N^1. This is

$$N^1 = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

The coset stabiliser:

$$N^{(1)} \geq \langle (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) \rangle.$$

The number of single cosets in [1] is $\frac{|N|}{|N^{(1)}|} = 13$, since the $|N^{(1)}|$=1. The orbits of $N^{(1)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and do right multiplication by the single coset representative Mt_1. We get the following:

$$Mt_1t_1 = M \in [*],$$

$$Mt_1t_2 \in [12],$$

$$Mt_1t_3 \in [13],$$

$$Mt_1t_4 \in [14],$$

$$t_1t_5 = x^{-1}t_9 \implies Mt_1t_5 = Mt_9 \in [1],$$

$$(since\ \{Nt_1)^n | n \in N\}\ and\ x^{-1} \in M),$$

$$t_1t_6 = x^7t_8t_3t_4t_2t_1t_3 \implies Mt_1t_6 = Mt_1t_3 \in [13],$$

$$(since\ \{N(t_1t_3)^n | n \in N\}\ and\ x^7t_8t_3t_4t_2 \in M)$$

$$Mt_1t_7 \in [17],$$

$$t_1t_8 = x^7t_8t_{13}t_{12}t_1t_4t_7$$

$$\implies Mt_1t_8 = Mt_4t_7 \in [14],$$
$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^7t_8t_{13}t_{12}t_1 \in M)$$
$$t_1t_9 = x^3t_3t_{12}t_{13}t_{12}t_1t_8t_{10}$$
$$\implies Mt_1t_9 = Mt_8t_{10} \in [13],$$
$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^3t_3t_{12}t_{13}t_{12}t_1 \in M)$$
$$t_1t_{10} = xt_6$$
$$\implies Mt_1t_{10} = Mt_6 \in [1],$$
$$(since\ \{N(t_1)^n|n \in N\}\ and\ x \in M)$$
$$t_1t_{11} = x^{10}t_{11}t_9t_{10}t_9t_{13}t_4t_{10}$$
$$\implies Mt_1t_{11} = Mt_4t_{10} \in [17],$$
$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^{10}t_{11}t_9t_{10}t_9t_{13} \in M)$$
$$t_1t_{12} = x^5t_8t_{10}$$
$$\implies Mt_1t_{12} = Mt_8t_{10} \in [13],$$
$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^5 \in M)$$
$$t_1t_{13} = x^{10}t_{11}t_{12}$$
$$\implies Mt_1t_{13} = Mt_{11}t_{12} \in [12].$$
$$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^{10} \in M)$$

After taking an element from each orbit and multiplying on the right, the new double cosets Mt_1t_2N, Mt_1t_3N, Mt_1t_4N and Mt_1t_7N with single coset representatives are Mt_1t_2, Mt_1t_3, Mt_1t_4 and Mt_1t_7. We represent them as [12], [13], [14], and [17], respectively. And the other double cosets are equal to other existing double cosets, we checked and proved which are equal.

$\boldsymbol{Mt_1t_2N}$

Looking back to our new double cosets which are [12],[13], [14] and [17]. We start first with the new double coset Mt_1t_2N by finding the coset stabilizer of $N^{(12)} = N^{12} = \rangle e \langle$. The order of $N^{(12)} = 1$, therefore the number of single cosets in Mt_1t_2N is $\frac{|N|}{|N^{(12)}|} = 13$. Lets find the orbits of $N^{(12)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$

which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Following the same procedure from above, take an element from each orbit and multiply on the right of the single coset Mt_1t_2 by the double coset Mt_1t_2N to get the following:

$t_1t_2t_1 \in [121]$,

$t_1t_2t_2 \in [1]$,

$t_1t_2t_3 = x^4t_4 \implies Mt_1t_2t_3 = Mt_4 \in [1]$,

$(since\ \{N(t_1)^n|n \in N\}\ and\ x^4 \in M)$

$t_1t_2t_4 = x^4t_6t_7t_6 \implies Mt_1t_2t_4 = Mt_6t_7t_6 \in [121]$,

$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^4 \in M)$

$t_1t_2t_5 \in [125]$,

$t_1t_2t_6 = x^4t_3t_9 \implies Mt_1t_2t_6 = Mt_3t_9 \in [17]$,

$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^4 \in M)$

$t_1t_2t_7 = x^5t_6t_1t_2t_{13}t_{10}t_{12} \implies Mt_1t_2t_7 = Mt_{10}t_{12} \in [13]$,

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^5t_6t_1t_2t_{13} \in M)$

$t_1t_2t_8 \in [128]$,

$t_1t_2t_9 = x^7t_7t_5t_6t_1t_8t_9t_2 \implies Mt_1t_2t_9 = Mt_8t_9t_2 \in [128]$,

$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^7t_7t_5t_6t_1 \in M)$

$t_1t_2t_{10} = x^9t_2t_3t_9 \implies Mt_1t_2t_{10} = Mt_2t_3t_9 \in [128]$,

$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^9 \in M)$

$t_1t_2t_{11} = x^6t_7t_{12}t_{11}t_{13}t_2t_4 \implies Mt_1t_2t_{11} = Mt_2t_4 \in [13]$,

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^6t_7t_{12}t_{11}t_{13} \in M)$

$t_1t_2t_{12} = x^5t_5t_8 \implies Mt_1t_2t_{12} = Mt_5t_8 \in [14]$,

$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^5 \in M)$

$$t_1t_2t_{13} = x^7t_6t_7t_{10} \implies Mt_1t_2t_{13} = Mt_6t_7t_{10} \in [125].$$
$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^7 \in M)$$

The new double coset representatives $Mt_1t_2t_1$, $Mt_1t_2t_5$, and $Mt_1t_2t_8$ denoted as [121], [125], and [128]. We needed to check which double cosets are equal to other existing double cosets. For example $Mt_1t_2t_3$N $= Mt_1N$, so one symmetric generator goes to [1]. Since t_3 is the only element on the orbit $\{3\}$. This is how we check which double cosets are equal to each other, and how many symmetric generators go to a different double cosets or they go back to itself.

$Mt_1t_2t_1N$

Continuing with the new double coset $Mt_1t_2t_1$, we find the coset stabilizer $N^{(121)} = N^{121}$=Identity. Only e will fix 1 and 2. Hence the number of single cosets in [121] is $\frac{|N|}{|N^{(121)}|} = \frac{13}{1} = 13$. The orbits of $N^{(121)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Take an element from each orbit and multiply on the right of the single coset representative $Mt_1t_2t_1$ by the double coset $Mt_1t_2t_1N$ to get the following double cosets, either new double cosets or cosets that are equal to each other.

$$t_1t_2t_1t_1 \in [12]$$
$$t_1t_2t_1t_2 = x^9t_9t_{11}t_{10}t_2t_{11}t_4 \implies Mt_1t_2t_1t_2 = Mt_{11}t_4 \in [17],$$
$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^9t_9t_{11}t_{10}t_2 \in M)$$
$$t_1t_2t_1t_3 = x^2t_{13}t_1t_4 \implies Mt_1t_2t_1t_3 = Mt_{13}t_1t_4 \in [125],$$
$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^2 \in M)$$
$$t_1t_2t_1t_4 = x^8t_9t_{11}t_8 \implies Mt_1t_2t_1t_4 = Mt_9t_{11}t_8 \in [1313],$$

$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^8 \in M)$$

$$t_1t_2t_1t_5 = x^8t_1t_2t_8 \implies Mt_1t_2t_1t_5 = Mt_1t_2t_8 \in [128],$$

$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^8 \in M)$$

$$t_1t_2t_1t_6 = x^5t_{10}t_1t_{13}t_7 \implies Mt_1t_2t_1t_6 = Mt_{10}t_{13}t_7 \in [1411],$$

$$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^5 \in M)$$

$$t_1t_2t_1t_7 = x^4t_6t_9t_3 \implies Mt_1t_2t_1t_7 = Mt_6t_9t_3 \in [1411],$$

$$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^4 \in M)$$

$$t_1t_2t_1t_8 = x^{10}t_{10}t_{11}t_4 \implies Mt_1t_2t_1t_8 = Mt_{10}t_{11}t_4 \in [128],$$

$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^{10} \in M)$$

$$t_1t_2t_1t_9 = x^{13}t_{11}t_{13}t_9 \implies Mt_1t_2t_1t_9 = Mt_{11}t_{13}t_9 \in [138],$$

$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^{13} \in M)$$

$$t_1t_2t_1t_{10} = xt_2t_3t_6 \implies Mt_1t_2t_1t_{10} = Mt_2t_3t_6 \in [125],$$

$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x \in M)$$

$$t_1t_2t_1t_{11} = x^3t_5t_8 \implies Mt_1t_2t_1t_{11} = Mt_5t_8 \in [14],$$

$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^3 \in M)$$

$$t_1t_2t_1t_{12} = x^9t_9t_{10} \implies Mt_1t_2t_1t_{12} = Mt_9t_{10} \in [12],$$

$$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^9 \in M)$$

$$t_1t_2t_1t_{13} = x^9t_{10}t_{12} \implies Mt_1t_2t_1t_{13} = Mt_{10}t_{12} \in [13].$$

$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^9 \in M)$$

From the work above there are no new double cosets, but we have to check and prove where those double cosets belong to. Since each set from the orbit has only one element, then there is only one symmetric generator go to different double cosets or back to itself.

$\boldsymbol{Mt_1t_2t_5N}$

The double coset $Mt_1t_2t_5$, we find the coset stabilizer $N^{(125)} = N^{125} = \langle e \rangle$. So the number of single cosets in [125] is $\frac{|N|}{|N^{(125)}|} = \frac{13}{1} = 13$ and the orbits of $N^{(13)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are the following:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and multiply on the right of the single coset $Mt_1t_2t_5$ by the double coset $Mt_1t_2t_5N$ to get the following:

$$t_1t_2t_5t_1 = x^8t_8t_{10}t_9t_1t_9t_{10}t_3 \implies Mt_1t_2t_5t_1 = Mt_9t_{10}t_3 \in [128],$$
$$t_1t_2t_5t_2 = x^7t_8t_{10}t_9t_1t_8t_{10}t_7 \implies Mt_1t_2t_5t_2 = Mt_8t_{10}t_7 \in [1313],$$
$$(since\ \{N(t_1t_3t_13)^n|n \in N\}\ and\ x^7t_8t_{10}t_9t_1 \in M)$$
$$t_1t_2t_5t_3 = x^2t_3t_{12}t_{13}t_{12}t_1t_2t_4t_1 \implies Mt_1t_2t_5t_3 = Mt_2t_4t_1 \in [1313],$$
$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^2t_3t_{12}t_{13}t_{12}t_1 \in M)$$
$$t_1t_2t_5t_4 = x^{11}t_2t_3t_2 \implies Mt_1t_2t_5t_4 = Mt_2t_3t_2 \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^{11} \in M)$$
$$t_1t_2t_5t_5 \in [12]$$
$$t_1t_2t_5t_6 = x^{11}t_1t_4 \implies Mt_1t_2t_5t_6 = Mt_1t_4 \in [14],$$
$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^{11} \in M)$$
$$t_1t_2t_5t_7 = x^8t_8t_{10}t_9t_1t_2t_8 \implies Mt_1t_2t_5t_7 = Mt_1t_2t_8 \in [17],$$
$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^8 \in M)$$
$$t_1t_2t_5t_8 = x^6t_9t_{10} \implies Mt_1t_2t_5t_8 = Mt_9t_{10} \in [12],$$
$$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^6 \in M)$$
$$t_1t_2t_5t_9 = x^{12}t_{13}t_1t_{13} \implies Mt_1t_2t_5t_9 = Mt_{13}t_1t_{13} \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^{12} \in M)$$
$$t_1t_2t_5t_{10} = x^5t_6t_1t_2t_{13}t_5t_7t_{12} \implies Mt_1t_2t_5t_{10} = Mt_5t_7t_{12} \in [138],$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^5t_6t_1t_2t_{13} \in M)$$
$$t_1t_2t_5t_{11} = x^{10}t_9t_{11}t_{10}t_2t_{12}t_1t_6 \implies Mt_1t_2t_5t_{11} = Mt_{12}t_1t_6 \in [138],$$
$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^{10}t_9t_{11}t_{10}t_2 \in M)$$
$$t_1t_2t_5t_{12} = x^9t_9t_{11}t_{10}t_2t_2t_3t_9 \implies Mt_1t_2t_5t_{12} = Mt_2t_3t_9 \in [128],$$
$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^9t_9t_{11}t_{10}t_2 \in M)$$
$$t_1t_2t_5t_{13} = x^3t_8t_{11}t_5 \implies Mt_1t_2t_5t_{13} = Mt_8t_{11}t_5 \in [1411].$$
$$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^3 \in M)$$

$\boldsymbol{Mt_1t_2t_8N}$

The last double coset from [12] is $Mt_1t_2t_8$ and its coset stabilizer $N^{(128)} = N^{128} = \langle e \rangle$. The number of single cosets in [128] is $\frac{|N|}{|N^{(128)}|} = \frac{13}{1} = 13$ and the orbits of $N^{(128)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and multiply on the right of the single coset $Mt_1t_2t_8$ by the double coset $Mt_1t_2t_8N$ to get the following:

$$t_1t_2t_8t_1 = x^3t_{12}t_1t_6 \implies Mt_1t_2t_8t_1 = Mt_{12}t_1t_6 \in [138],$$
$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^3 \in M)$$
$$t_1t_2t_8t_2 = x^7t_8t_{13}t_{12}t_1t_7t_8 \implies Mt_1t_2t_8t_2 = Mt_7t_8 \in [12],$$
$$(since\ \{N(t_1t_2)^n | n \in N\}\ and\ x^7t_8t_{13}t_{12}t_1 \in M)$$
$$t_1t_2t_8t_3 = x^{10}t_{12}t_1t_{11} \implies Mt_1t_2t_8t_3 = Mt_{12}t_1t_{11} \in [1313],$$
$$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^{10} \in M)$$
$$t_1t_2t_8t_4 = x^6t_6t_9 \implies Mt_1t_2t_8t_4 = Mt_6t_9 \in [14],$$
$$(since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^6 \in M)$$
$$t_1t_2t_8t_5 = x^5t_1t_2t_1 \implies Mt_1t_2t_8t_5 = Mt_1t_2t_1 \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x^5 \in M)$$
$$t_1t_2t_8t_6 = x^6t_7t_{12}t_{11}t_{13}t_6t_7t_{10} \implies Mt_1t_2t_8t_6 = Mt_6t_7t_{10} \in [125],$$
$$(since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ x^6t_7t_{12}t_{11}t_{13} \in M)$$
$$t_1t_2t_8t_7 = x^8t_9t_{11}t_{10}t_2t_2t_4t_1 \implies Mt_1t_2t_8t_7 = Mt_2t_4t_1 \in [1313],$$
$$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^8t_9t_{11}t_{10}t_2 \in M)$$
$$t_1t_2t_8t_8 \in [12]$$
$$t_1t_2t_8t_9 = x^4t_{13}t_1 \implies Mt_1t_2t_8t_9 = Mt_{13}t_1 \in [12],$$
$$(since\ \{N(t_1t_2)^n | n \in N\}\ and\ x^4 \in M)$$
$$t_1t_2t_8t_{10} = x^6t_6t_4t_5t_{13}t_9t_{11}t_3 \implies Mt_1t_2t_8t_{10} = Mt_9t_{11}t_3 \in [138],$$

$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^6t_6t_4t_5t_{13} \in M)$$

$$t_1t_2t_8t_{11} = x^4t_5t_{13}t_1t_{12}t_{13}t_1t_4 \implies Mt_1t_2t_8t_{11} = Mt_{13}t_1t_4 \in [125],$$

$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^4t_5t_{13}t_1t_{12} \in M)$$

$$t_1t_2t_8t_{12} = x^3t_5t_6t_5 \implies Mt_1t_2t_8t_{12} = Mt_5t_6t_5 \in [121],$$

$$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^3 \in M)$$

$$t_1t_2t_8t_{13} = x^9t_8t_{10}t_9t_1t_1t_7 \implies Mt_1t_2t_8t_{13} = Mt_1t_7 \in [17].$$

$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^9t_8t_{10}t_9t_1 \in M)$$

Mt_1t_3N

Now the new double coset Mt_1t_3N, we find the coset stabilizer $N^{(13)} = N^{13} = \langle e \rangle$. Only identity will fix 1 and 3. Hence the number of single cosets in [13] is $\frac{|N|}{|N^{(13)}|} = \frac{13}{1} = 13$ and the orbits of $N^{(13)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and multiply on the right of the single coset Mt_1t_3 by the double coset Mt_1t_3N to get the following:

$$t_1t_3t_1 = x^9t_8t_{10} \implies Mt_1t_3t_1 = Mt_8t_{10} \in [13],$$

$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^9 \in M)$$

$$t_1t_3t_2 = x^{10}t_9t_{11}t_{10}t_2t_7 \implies Mt_1t_3t_2 = Mt_7 \in [1],$$

$$(since\ \{N(t_1)^n|n \in N\}\ and\ x^{10}t_9t_{11}t_{10}t_2 \in M)$$

$$t_1t_3t_3 \in [1]$$

$$t_1t_3t_4 = x^4t_5t_6t_5 \implies Mt_1t_3t_4 = Mt_5t_6t_5 \in [121],$$

$$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^4 \in M)$$

$$t_1t_3t_5 = x^8t_7 \implies Mt_1t_3t_5 = Mt_7 \in [1],$$

$$(since\ \{N(t_1)^n|n \in N\}\ and\ x^8 \in M)$$

$$t_1t_3t_6 = x^6t_8t_{10}t_9t_1t_1 \implies Mt_1t_3t_6 = Mt_1 \in [1],$$

$(since\ \{N(t_1)^n|n \in N\}\ and\ x^6t_8t_{10}t_9t_1 \in M)$

$t_1t_3t_7 = x^4t_7t_9 \implies Mt_1t_3t_7 = Mt_7t_9 \in [13],$

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^4 \in M)$

$t_1t_3t_8 \in [138]$

$t_1t_3t_9 = x^7t_8t_3t_4t_2t_9t_{12} \implies Mt_1t_3t_9 = Mt_9t_{12} \in [13],$

$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^7t_8t_3t_4t_2 \in M)$

$t_1t_3t_{10} = x^{11}t_{12}t_{10}t_{11}t_{10}t_1t_{13}t_1 \implies Mt_1t_3t_{10} = Mt_{13}t_1 \in [12],$

$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^{11}t_{12}t_{10}t_{11}t_{10}t_1 \in M)$

$t_1t_3t_{11} = x^{11}t_{12}t_{10}t_{11}t_{10}t_1t_5t_6 \implies Mt_1t_3t_{11} = Mt_5t_6 \in [12],$

$(since\ \{N(t_1t_2)^n|n \in N\}\ and\ x^{11}t_{12}t_{10}t_{11}t_{10}t_1 \in M)$

$t_1t_3t_{12} = xt_2t_8 \implies Mt_1t_3t_{12} = Mt_2t_8 \in [17].$

$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x \in M)$

$t_1t_3t_{13} \in [1313]$

The new double coset representatives $Mt_1t_3t_8$ and $Mt_1t_3t_{13}$ denoted as [138], [1313]. We need to check which double cosets are equal to other existing double cosets. For example $Mt_1t_3t_1$N $= Mx^9t_1t_3N$, so one symmetric generator goes to [13]. Since t_1 is the only element on the orbit $\{1\}$. This is how we check which double cosets are equal to each other, and how many symmetric generators go to a different double cosets or they go back to itself.

$\boldsymbol{Mt_1t_3t_8N}$

Continuing with the new double coset $Mt_1t_3t_8N$, we find the coset stabilizer $N^{(138)} = N^{138} = \langle e \rangle$. Only identity will fix 1, 3, and 8.The number of single cosets in [138] is $\frac{|N|}{|N^{(13)}|} = \frac{13}{1} = 13$ and the orbits of $N^{(138)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and multiply on the right of the single coset $Mt_1t_3t_8$ by the double coset $Mt_1t_3t_8N$ to get the following:

$t_1t_3t_8t_1 = x^5t_6t_4t_5t_{13}t_8t_{11}t_5 \implies Mt_1t_3t_8t_1 = Mt_8t_{11}t_5 \in [1411],$

$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^5t_6t_4t_5t_{13} \in M)$

$t_1t_3t_8t_2 = x^8t_7t_{12}t_{11}t_{13}t_6t_7t_{13} \implies Mt_1t_3t_8t_2 = Mt_6t_7t_{13} \in [128],$

$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^8t_7t_{12}t_{11}t_{13} \in M)$

$t_1t_3t_8t_3 = x^{10}t_3t_4t_{10} \implies Mt_1t_3t_8t_3 = Mt_3t_4t_{10} \in [128],$

$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^{10} \in M)$

$t_1t_3t_8t_4 = x^2t_3t_9 \implies Mt_1t_3t_8t_4 = Mt_3t_9 \in [17],$

$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^2 \in M)$

$t_1t_3t_8t_5 = Identity t_{11}t_1t_8 \implies Mt_1t_3t_8t_5 = Mt_{11}t_1t_8 \in [1411],$

$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ Identity \in M)$

$t_1t_3t_8t_6 = x^{10}t_{11}t_9t_{10}t_9t_{13}t_{10}t_{11}t_1 \implies Mt_1t_3t_8t_6 = Mt_{10}t_{11}t_1 \in [125],$

$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^{10}t_{11}t_9t_{10}t_9t_{13} \in M)$

$t_1t_3t_8t_7 = x^{10}t_{11}t_9t_{10}t_9t_{13}t_{10}t_{12}t_4 \implies Mt_1t_3t_8t_7 = Mt_{10}t_{12}t_4 \in [138],$

$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^{10}t_{11}t_9t_{10}t_9t_{13} \in M)$

$t_1t_3t_8t_8 \in [13]$

$t_1t_3t_8t_9 = x^8t_{11}t_{13}t_{10} \implies Mt_1t_3t_8t_9 = Mt_{11}t_{13}t_{10} \in [1313],$

$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^8 \in M)$

$t_1t_3t_8t_{10} = x^6t_7t_5t_6t_1t_9t_{12} \implies Mt_1t_3t_8t_{10} = Mt_9t_{12} \in [14],$

$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^6t_7t_5t_6t_1 \in M)$

$t_1t_3t_8t_{11} = x^5t_6t_1t_2t_{13}t_5t_7t_{12} \implies Mt_1t_3t_8t_{11} = Mt_5t_7t_{12} \in [138],$

$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^5t_6t_1t_2t_{13} \in M)$

$t_1t_3t_8t_{12} = x^3t_4t_5t_4 \implies Mt_1t_3t_8t_{12} = Mt_4t_5t_4 \in [121],$

$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^3 \in M)$

$t_1t_3t_8t_{13} = x^2t_3t_{12}t_{13}t_{12}t_1t_3t_4t_7 \implies Mt_1t_3t_8t_{13} = Mt_3t_4t_7 \in [125].$

$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^2t_3t_{12}t_{13}t_{12}t_1 \in M)$

$Mt_1t_3t_{13}N$

Now the double coset $Mt_1t_3t_{13}N$, we find the coset stabilizer $N^{(1313)} = N^{1313}$ $= \langle e \rangle$. Only identity will fix 1, 3, and 13.The number of single cosets in [1313] is $\frac{|N|}{|N^{(13)}|} = \frac{13}{1} = 13$ and the orbits of $N^{(1313)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Now take an element from each orbit and multiply on the right of the single coset $Mt_1t_3t_{13}$ by the double coset $Mt_1t_3t_{13}N$ to get the following:

$$t_1t_3t_{13}t_1 = x^7t_8t_{10}t_9t_1t_5t_7t_4 \implies Mt_1t_3t_{13}t_1 = Mt_5t_7t_4 \in [1313],$$
$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^7t_8t_{10}t_9t_1 \in M)$$
$$t_1t_3t_{13}t_2 = x^5t_6t_4t_5t_{13}t_{13}t_1t_4 \implies Mt_1t_3t_{13}t_2 = Mt_{13}t_1t_4 \in [125],$$
$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^5t_6t_4t_5t_{13} \in M)$$
$$t_1t_3t_{13}t_3 = x^4t_{13}t_3t_{10} \implies Mt_1t_3t_{13}t_3 = Mt_{13}t_3t_{10} \in [1411],$$
$$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^4 \in M)$$
$$t_1t_3t_{13}t_4 = x^6t_7t_2t_3t_1t_8t_{11} \implies Mt_1t_3t_{13}t_4 = Mt_8t_{11} \in [14],$$
$$(since\ \{N(t_1t_4)^n|n \in N\}\ and\ x^6t_7t_2t_3t_1 \in M)$$
$$t_1t_3t_{13}t_5 = x^3t_3t_4t_{10} \implies Mt_1t_3t_{13}t_5 = Mt_3t_4t_{10} \in [128],$$
$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^3 \in M)$$
$$t_1t_3t_{13}t_6 = x^5t_6t_1t_2t_{13}t_{13}t_1t_7 \implies Mt_1t_3t_{13}t_6 = Mt_{13}t_1t_7 \in [128],$$
$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^5t_6t_1t_2t_{13} \in M)$$
$$t_1t_3t_{13}t_7 = x^7t_7t_5t_6t_1t_3t_6t_{13} \implies Mt_1t_3t_{13}t_7 = Mt_3t_6t_{13} \in [1411],$$
$$(since\ \{N(t_1t_4t_{11})^n|n \in N\}\ and\ x^7t_7t_5t_6t_1 \in M)$$
$$t_1t_3t_{13}t_8 = x^7t_8t_3t_4t_2t_7t_8t_{11} \implies Mt_1t_3t_{13}t_8 = Mt_7t_8t_{11} \in [125],$$
$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^7t_8t_3t_4t_2 \in M)$$
$$t_1t_3t_{13}t_9 = x^5t_6t_7t_6 \implies Mt_1t_3t_{13}t_9 = Mt_6t_7t_6 \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n|n \in N\}\ and\ x^5 \in M)$$
$$t_1t_3t_{13}t_{10} = x^8t_7t_{12}t_{11}t_{13}t_{10}t_{12}t_9 \implies Mt_1t_3t_{13}t_{10} = Mt_{10}t_{12}t_9 \in [1313],$$

$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^8t_7t_{12}t_{11}t_{13} \in M)$$

$$t_1t_3t_{13}t_{11} = x^6t_7t_2t_3t_1t_2t_8 \implies Mt_1t_3t_{13}t_{11} = Mt_2t_8 \in [17],$$

$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^6t_7t_2t_3t_1 \in M)$$

$$t_1t_3t_{13}t_{12} = x^5t_4t_6t_{11} \implies Mt_1t_3t_{13}t_{12} = Mt_4t_6t_{11} \in [138].$$

$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^5 \in M)t_1t_3t_{13}t_{13} \qquad \in [13]$$

Mt_1t_4N

Now the new double coset Mt_1t_4, the coset stabilizer $N^{(14)} = N^{14}$=Identity. Only e will fix 1 and 4. Hence the number of single cosets in [14] is $\frac{|N|}{|N^{(14)}|} = \frac{13}{1} = 13$. The orbits of $N^{(14)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Take an element from each orbit and multiply on the right of the single coset representative Mt_1t_4 by the double coset Mt_1t_4N to get the following double cosets, either new double cosets or cosets that are equal to each other.

$$t_1t_4t_1 = x^{11}t_{12}t_{10}t_{11}t_{10}t_1t_6t_8 \implies Mt_1t_4t_1 = Mt_6t_8 \in [13],$$

$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^{11}t_{12}t_{10}t_{11}t_{10}t_1 \in M)$$

$$t_1t_4t_2 = x^8t_8t_{13}t_{12}t_1t_6t_8t_{13} \implies Mt_1t_4t_2 = Mt_6t_8t_{13} \in [138],$$

$$(since\ \{N(t_1t_3t_8)^n|n \in N\}\ and\ x^8t_8t_{13}t_{12}t_1 \in M)$$

$$t_1t_4t_3 = x^{11}t_{12}t_{10}t_{11}t_{10}t_1t_8t_1 \implies Mt_1t_4t_3 = Mt_8t_1 \in [17],$$

$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^{11}t_{12}t_{10}t_{11}t_{10}t_1 \in M)$$

$$t_1t_4t_4 \in [1]$$

$$t_1t_4t_5 = x^9t_9t_{11}t_{10}t_2t_{11} \implies Mt_1t_4t_5 = Mt_{11} \in [1],$$

$$(since\ \{N(t_1)^n|n \in N\}\ and\ x^9t_9t_{11}t_{10}t_2 \in M)$$

$$t_1t_4t_6 = x^2t_1t_2t_5 \implies Mt_1t_4t_6 = Mt_1t_2t_5 \in [125],$$

$$(since\ \{N(t_1t_2t_5)^n|n \in N\}\ and\ x^2 \in M)$$

$$t_1t_4t_7 = x^{10}t_{10}t_{11}t_{10} \implies Mt_1t_4t_7 = Mt_{10}t_{11}t_{10} \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x^{10} \in M)$$
$$t_1t_4t_8 = x^8t_{10}t_{11} \implies Mt_1t_4t_8 = Mt_{10}t_{11} \in [12],$$
$$(since\ \{N(t_1t_2)^n | n \in N\}\ and\ x^8 \in M)$$
$$t_1t_4t_9 = x^8t_8t_{10}t_9t_1t_{10}t_{13} \implies Mt_1t_4t_9 = Mt_{10}t_{13} \in [14],$$
$$(since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^8t_8t_{10}t_9t_1 \in M)$$
$$t_1t_4t_{10} = x^6t_8t_{10}t_9t_1t_7t_9t_6 \implies Mt_1t_4t_{10} = Mt_7t_9t_6 \in [1313],$$
$$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^6t_8t_{10}t_9t_1 \in M)$$
$$t_1t_4t_{11} \in [1411]$$
$$t_1t_4t_{12} = x^7t_9t_{10}t_3 \implies Mt_1t_4t_{12} = Mt_9t_{10}t_3 \in [128],$$
$$(since\ \{N(t_1t_2t_8)^n | n \in N\}\ and\ x^7 \in M)$$
$$t_1t_4t_{13} = x^2t_3t_{12}t_{13}t_{12}t_1t_5t_8 \implies Mt_1t_4t_{13} = Mt_5t_8 \in [14].$$
$$(since\ \{N(t_1t_2t_8)^n | n \in N\}\ and\ x^2t_3t_{12}t_{13}t_{12}t_1 \in M)$$

After taking an element from each orbit and multiplying on the right, the new double cosets with single coset representatives is $Mt_1t_4t_{11}$. We represent it as [1411], respectively. And the other double cosets are equal to other existing double cosets, so we checked and proved which are equal.

$\boldsymbol{Mt_1t_4t_{11}N}$

Continuing with the new double coset $Mt_1t_4t_{11}$, the coset stabilizer $N^{(1411)} = N^{1411}$=Identity. Only e will fix 1, 4, and 11. Hence the number of single cosets in [1411] is $\frac{|N|}{|N^{(1411)}|} = \frac{13}{1} = 13$. The orbits of $N^{(1411)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Take an element from each orbit and multiply on the right of the single coset representative $Mt_1t_4t_{11}$ by the double coset $Mt_1t_4t_{11}N$. We have:

$t_1t_4t_{11}t_1 = x^9t_9t_{11}t_{10}t_2t_1t_7 \implies Mt_1t_4t_{11}t_1 = Mt_1t_7 \in [17],$

$(since\ \{N(t_1t_7)^n | n \in N\}\ and\ x^9t_9t_{11}t_{10}t_2 \in M)$

$t_1t_4t_{11}t_2 = x^9t_9t_{10}t_9 \implies Mt_1t_4t_{11}t_2 = Mt_9t_{10}t_9 \in [121],$

$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x^9 \in M)$

$t_1t_4t_{11}t_3 = x^8t_7t_{12}t_{11}t_{13}t_5t_8t_2 \implies Mt_1t_4t_{11}t_3 = Mt_5t_8t_2 \in [1411],$

$(since\ \{N(t_1t_4t_{11})^n | n \in N\}\ and\ x^8t_7t_{12}t_{11}t_{13} \in M)$

$t_1t_4t_{11}t_4 = x^9t_2t_4t_1 \implies Mt_1t_4t_{11}t_4 = Mt_2t_4t_1 \in [1313],$

$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^9 \in M)$

$t_1t_4t_{11}t_5 = x^8t_8t_{13}t_{12}t_1t_{12}t_1t_{11} \implies Mt_1t_4t_{11}t_5 = Mt_{12}t_1t_{11} \in [1313],$

$(since\ \{N(t_1t_3t_{13})^n | n \in N\}\ and\ x^8t_8t_{13}t_{12}t_1 \in M)$

$t_1t_4t_{11}t_6 = x^{10}t_7t_8t_{11} \implies Mt_1t_4t_{11}t_6 = Mt_7t_8t_{11} \in [125],$

$(since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ x^{10} \in M)$

$t_1t_4t_{11}t_7 = x^3t_3t_{12}t_{13}t_{12}t_1t_7t_9t_1 \implies Mt_1t_4t_{11}t_7 = Mt_7t_9t_1 \in [138],$

$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^3t_3t_{12}t_{13}t_{12}t_1 \in M)$

$t_1t_4t_{11}t_8 = Identity t_4t_6t_4 \implies Mt_1t_4t_{11}t_8 = Mt_4t_6t_4 \in [138],$

$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ Identity \in M)$

$t_1t_4t_{11}t_9 = x^9t_8t_{13}t_{12}t_1t_{10}t_{13}t_7 \implies Mt_1t_4t_{11}t_9 = Mt_{10}t_{13}t_7 \in [1411],$

$(since\ \{N(t_1t_4t_{11})^n | n \in N\}\ and\ x^9t_8t_{13}t_{12}t_1 \in M)$

$t_1t_4t_{11}t_{10} = x^8t_5t_6t_5 \implies Mt_1t_4t_{11}t_{10} = Mt_5t_6t_5 \in [121],$

$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x^8 \in M)$

$t_1t_4t_{11}t_{11} \in [14]$

$t_1t_4t_{11}t_{12} = x^6t_7t_5t_6t_1t_{10}t_{13}t_7 \implies Mt_1t_4t_{11}t_{12} = Mt_{10}t_{13}t_7 \in [1411],$

$(since\ \{N(t_1t_4t_{11})^n | n \in N\}\ and\ x^6t_7t_5t_6t_1 \in M)$

$t_1t_4t_{11}t_{13} = x^6t_8t_{10}t_9t_1t_5t_8t_2 \implies Mt_1t_4t_{11}t_{13} = Mt_5t_8t_2 \in [1411].$

$(since\ \{N(t_1t_4t_{11})^n | n \in N\}\ and\ x^6t_8t_{10}t_9t_1 \in M)$

Mt_1t_7N

Now the new double coset Mt_1t_7, the coset stabilizer $N^{(17)} = N^{17}$=Identity. Only e will fix 1 and 7. Hence the number of single cosets in [17] is $\frac{|N|}{|N^{(17)}|} = \frac{13}{1} = 13$. The orbits of $N^{(17)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}.$$

Take an element from each orbit and multiply on the right of the single coset representative Mt_1t_7 by the double coset Mt_1t_7N to get the following double cosets, either new double cosets or cosets that are equal to each other.

$$t_1t_7t_1 = x^9t_8t_{13}t_{12}t_1t_4t_{11} \implies Mt_1t_7t_1 = Mt_1t_4t_{11} \in [1411],$$
$$(since\ \{N(t_1t_4t_{11})^n | n \in N\}\ and\ x^9t_8t_{13}t_{12}t_1 \in M)$$
$$t_1t_7t_2 = x^{11}t_{12}t_1t_6 \implies Mt_1t_7t_2 = Mt_{12}t_1t_6 \in [138],$$
$$(since\ \{N(t_1t_3t_8)^n | n \in N\}\ and\ x^{11} \in M)$$
$$t_1t_7t_3 = x^{11}t_{12}t_{10}t_{11}t_{10}t_1t_5t_{11} \implies Mt_1t_7t_3 = Mt_5t_{11} \in [17],$$
$$(since\ \{N(t_1t_7)^n | n \in N\}\ and\ x^{11}t_{12}t_{10}t_{11}t_{10}t_1 \in M)$$
$$t_1t_7t_4 = x^7t_8t_3t_4t_2t_{12}t_{13} \implies Mt_1t_7t_4 = Mt_{12}t_{13} \in [12],$$
$$(since\ \{N(t_1t_2)^n | n \in N\}\ and\ x^7t_8t_3t_4t_2 \in M)$$
$$t_1t_7t_5 = x^7t_8t_{13}t_{12}t_1t_4t_5t_4 \implies Mt_1t_7t_5 = Mt_4t_5t_4 \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x^7t_8t_{13}t_{12}t_1 \in M)$$
$$t_1t_7t_6 = x^7t_8t_3t_4t_2t_{13}t_1t_4 \implies Mt_1t_7t_6 = Mt_{13}t_1t_4 \in [125],$$
$$(since\ \{N(t_1t_2t_5)^n | n \in N\}\ and\ x^7t_8t_3t_4t_2 \in M)$$
$$t_1t_7t_7 \in [1]$$
$$t_1t_7t_8 = x^6t_7t_2t_3t_1t_{11} \implies Mt_1t_7t_8 = Mt_{11} \in [1],$$
$$(since\ \{N(t_1)^n | n \in N\}\ and\ x^6t_7t_2t_3t_1 \in M)$$
$$t_1t_7t_9 = x^7t_7t_{12}t_{11}t_{13}t_7t_{10} \implies Mt_1t_7t_9 = Mt_7t_{10} \in [14],$$
$$(since\ \{N(t_1t_4)^n | n \in N\}\ and\ x^7t_7t_{12}t_{11}t_{13} \in M)$$
$$t_1t_7t_{10} = x^7t_5t_6t_1t_{13}t_2t_{12} \implies Mt_1t_7t_{10} = Mt_{13}t_2t_{12} \in [1313],$$

$$(since\ \{N(t_1t_3t_{13})^n|n \in N\}\ and\ x^7t_5t_6t_1 \in M)$$

$$t_1t_7t_{11} = x^{12}t_{13}t_2 \implies Mt_1t_7t_{11} = Mt_{13}t_2 \in [13],$$

$$(since\ \{N(t_1t_3)^n|n \in N\}\ and\ x^{12} \in M)$$

$$t_1t_7t_{12} = x^5t_6t_1t_2t_{13}t_{10}t_3 \implies Mt_1t_7t_{12} = Mt_{10}t_3 \in [17],$$

$$(since\ \{N(t_1t_7)^n|n \in N\}\ and\ x^5t_6t_1t_2t_{13} \in M)$$

$$t_1t_7t_{13} = x^4t_5t_{13}t_1t_{12}t_1t_2t_8 \implies Mt_1t_7t_{13} = Mt_1t_2t_8 \in [128].$$

$$(since\ \{N(t_1t_2t_8)^n|n \in N\}\ and\ x^4t_5t_{13}t_1t_{12} \in M)$$

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, since the index of M in G is 144 . We conclude:

$$G = MeN \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_2t_1N \cup Mt_1t_2t_5N \cup Mt_1t_2t_8N \cup Mt_1t_3N \\ \cup Mt_1t_3t_8N \cup Mt_1t_3t_{13}N \cup Mt_1t_4N \cup Mt_1t_4t_{11}N \cup Mt_1t_7N$$

where

$$G = \frac{2^{*13} : 13}{(x^{-4}t)^3, (x^2t)^4, (x^{-1}t)^4}$$

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(128)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(138)}|} \\ + \frac{|N|}{|N^{(1313)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(1411)}|} + \frac{|N|}{|N^{(17)}|} \times |M|$$

$|G| \leq (1 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13) \times 39$

$|G| \leq 144 \times 39$

$|G| \leq 5616.$

The Cayley diagram summarizes the information listed above.

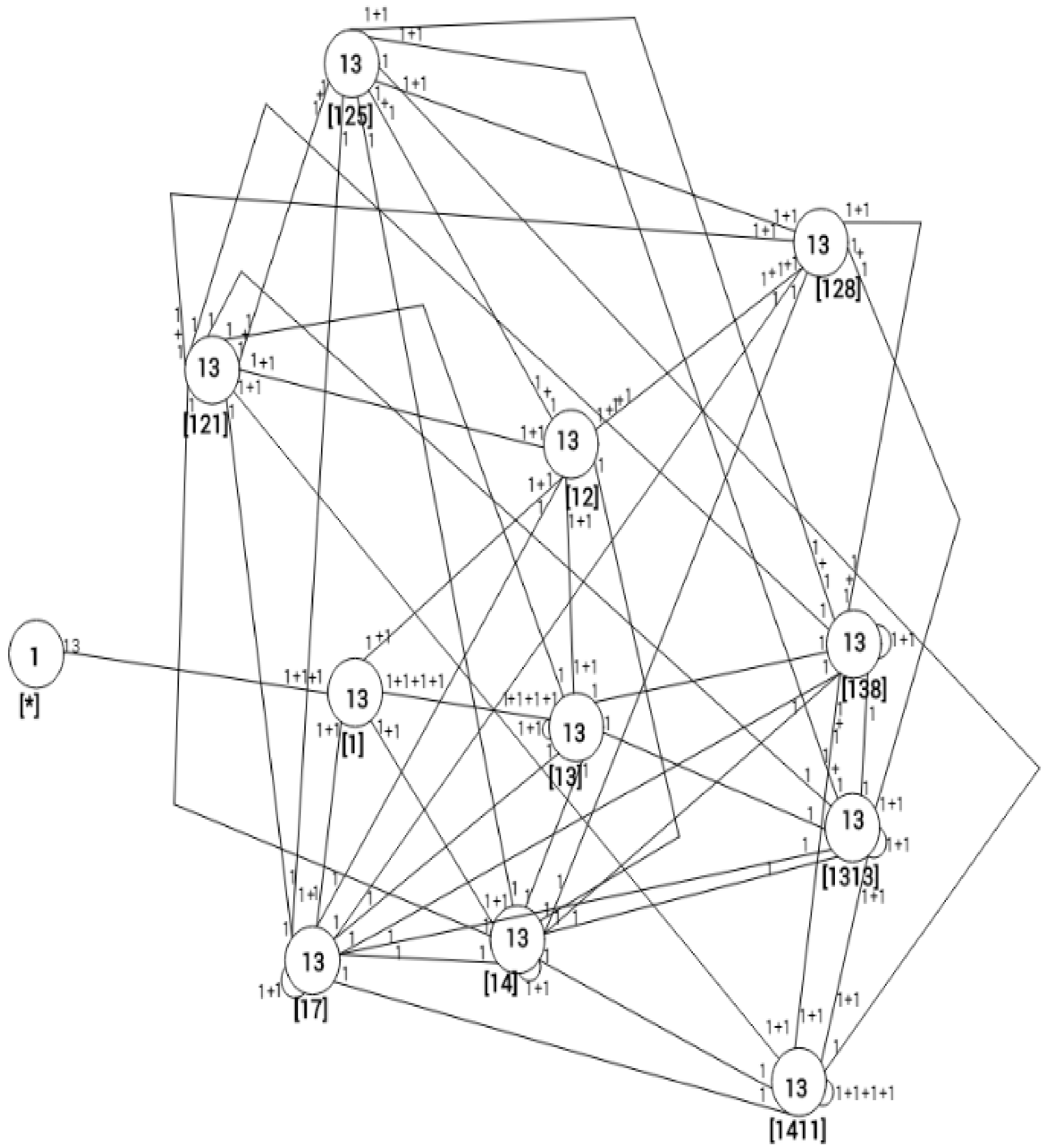

Figure 6.2: Cayley graph of $PSL_3(3)$ over $M = (13:3)$

6.3 Double Coset Enumeration over a Maximal Subgroup

Using the process of maximal subgroup on a different example. In this example we have to do a few changes to N. The control group of $2^{*12} : (S_4 \times 2)$ was of kernel

2. Hence, to have a better image we changed the kernel to be 1. Thus we were able to continue by constructing a double coset enumeration of $PSL_2(8)$ over a maximal subgroup with the new progenitor $2^{*3} : S_3$.

6.3.1 Construct of $PSL_2(8)$ over $M = (9 : 2)$

Before we apply double coset enumeration over maximal subgroup we are going to start by factoring the progenitor, $2^{*3} : (S_3)$, by the relations $(xt)^7$, $(xyt^x)^2$, and $(yt)^9$, denoted as

$$\langle x, y, t | x^2, y^3, (y^{-1}x)^2,$$

$$t^2,$$

$$(t, xy^{-1}), (xt)^7, (xyt^x)^2, (yt)^9 >;$$

and $N \cong S_3 =$

$$\langle x, y | x^2, y^3, (y^{-1}x)^2 \rangle.$$

So we obtain the following homomorphic image:

$$G \cong \frac{2^{*3} : (S_3)}{(xt)^7, (xyt^x)^2, (yt)^9} \cong PSL_2(8),$$

where $x \sim (1, 2)$, $y \sim (1, 2, 3)$, and $t \sim t_1$.
Now expand the first relator $(xt)^7 = 1$ to:
$(xt)^7 = 1$, which it can be expand:

$$\begin{aligned} (xt)^7 &= 1 \\ x^7 t_1^{x^6} t_1^{x^5} t_1^{x^4} t_1^{x^3} t_1^{x^2} t_1^{x} t_1 &= 1 \\ (1, 2) t_1 t_2 t_1 t_2 t_1 t_2 t_1 &= 1 \\ t_1 t_2 t_1 &= (1, 2) t_1 t_2 t_1 t_2 \end{aligned}$$

After expanding the second relation $(xyt^x)^2 = e$ we noticed that this relation simplifies to identity. Thus we continue with the third relation.

And the third relation can be expand as follow:

$$\begin{aligned}(yt)^9 &= 1\\ y^9 t_1^{y^8} t_1^{y^7} t_1^{y^6} t_1^{y^5} t_1^{y^4} t_1^{y^3} t_1^{y^2} t_1^{y} t_1 &= 1\\ t_3t_2t_1t_3t_2t_1t_3t_2t_1 &= 1\\ t_3t_2t_1t_3 &= t_1t_2t_3t_1t_2\end{aligned}$$

Let M be the maximal subgroup generated by the control group $N = S_3$ and $t_3t_2t_1t_2t_3t_2t_1 = t^{y^2}t^x tt^x t^{y^2} t^x t$

$$M = \langle < x, y >, t^{y^2}t^x tt^x t^{y^2} t^x t\rangle, = (9 : 2) \text{ where } |M| = 18.$$

Therefore, M is the maximal subgroup.

Let us start constructing a manual double coset enumeration of G over the maximal subgroup, M and N. Denote $[w]$ to be the double coset MwN, where w is a word in $t_i's$.

MeN

We begin with the double coset MeN, denote $[*]$. This double coset contains only one single coset, namely M. The single coset stabilizer of M is N, which is transitive on $\{t_1, t_2, t_3\}$ and therefore, has a single orbit,

$$\mathcal{O} = \{1, 2, 3\}.$$

Now take an element from $\mathcal{O}$ say t_1 and multiply the single coset representative M by t_1 to obtain Mt_1. This is a new double coset Mt_1N, denote as [1], so three symmetric generators will go to our new double coset, [1].

$\boldsymbol{Mt_1N}$

Continuing with the double coset Mt_1N, we find the point stabilizer N^1. This is

$$N^1 = \{1\}, \{2\}, \{3\}.$$

But, the coset stabiliser of N^1 is:

$$N^{(1)} \geq \langle 1, (2,3) \rangle.$$

Thus the number of single cosets in [1] is $\frac{|N|}{|N^{(1)}|} = 3$, since the $|N^{(1)}|$=2. The orbits of $N^{(1)}$ on $\{t_1, t_2, t_3\}$ are:

$$\mathcal{O} = \{1\}, \{2, 3\}.$$

Now take an element from each orbit and right multiply to the single coset representative Mt_1. We get the following:

$$Mt_1t_1 = M \in [*],$$
$$Mt_1t_2 \in [12]$$

After taking an element from each orbit and right multiply to the single coset representative, Mt_1 we get a new double coset Mt_1t_2N with single coset representative is Mt_1t_2. We represent Mt_1t_2N as [12] respectively.

$\boldsymbol{Mt_1t_2N}$

The new double coset Mt_1t_2N and the coset stabilizer of $N^{(12)}$ is $N^{12} = \langle e \rangle$. So the order of $N^{(12)} is 1$, therefore the number of single cosets in Mt_1t_2N is $\frac{|N|}{|N^{(12)}|} = 6$. Now find the orbits of $N^{(12)}$ on $\{t_1, t_2, t_3\}$ which are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}.$$

Following the same process from above, take an element from each orbit and multiply on the right of the single coset representative Mt_1t_2 of the double coset Mt_1t_2N to get the following:

$$Mt_1t_2t_1 \in [121],$$
$$Mt_1t_2t_2 \in [1],$$
$$Mt_1t_2t_3 \in [123]$$

Hence, there are two new double coset $Mt_1t_2t_1N$ and $Mt_1t_2t_3N$ with the single coset representatives $Mt_1t_2t_1$ and $Mt_1t_2t_3$, denoted by [121] and [123]. Note, the third single coset is equal to an existing double coset [1], since $Mt_1t_2t_2$N $= Mt_1N$ where t_2t_2 is identity. Thus, one symmetric generator goes back to [1]. Throughout the process we are going to use the same method to check and verify where does the single cosets equals to or if they loop back to itself.

$\boldsymbol{Mt_1t_2t_1N}$

Continuing with the new double coset $Mt_1t_2t_1$, we find the coset stabilizer $N^{(121)} = N^{121}$=Identity. Only e will fix 1 and 2. Hence the number of single cosets in [121] is $\frac{|N|}{|N^{(121)}|} = \frac{6}{1} = 6$. The orbits of $N^{(121)}$ on $\{t_1, t_2, t_3\}$ are:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}.$$

Take an element from each orbit and right multiply to the single coset representative $Mt_1t_2t_1$ of the double coset $Mt_1t_2t_1N$. We check and verify if there are any new double cosets or if there are double cosets that are either equal to other existing double cosets or if the double coset loops back to itself.

$$Mt_1t_2t_1t_1 = Mt_1t_2 \in [12]$$
$$t_1t_2t_1t_2 = xt_1t_2t_1 \implies Mt_1t_2t_1t_2 = Mt_1t_2t_1 \in [121],$$
$$(since\ \{N(t_1t_2t_1)^n | n \in N\}\ and\ x \in M)$$
$$Mt_1t_2t_1t_3 \in [1213]$$

From above we have a new double coset $Mt_1t_2t_1t_3N$ with single coset representative

$Mt_1t_2t_1t_3$, denoted as [1213]. The other two double cosets are equal to existing double cosets Mt_1t_2N and $Mt_1t_2t_1N$.

$\boldsymbol{Mt_1t_2t_3N}$

The double coset $Mt_1t_2t_3$, we find the coset stabilizer $N^{(123)} = N^{123} =< e >$. So the number of single cosets in [123] is $\frac{|N|}{|N^{(123)}|} = \frac{6}{1} = 6$ and the orbits of $N^{(123)}$ on $\{t_1, t_2, t_3\}$ are the following:

$$\mathcal{O} = \{1\}, \{2\}, \{3\}.$$

Now take an element from each orbit and right multiply of the single coset $Mt_1t_2t_3$ by the double coset $Mt_1t_2t_3N$ to get the following:

$$t_1t_2t_3t_1 = xy^2xy^2t_1t_2t_3t_1t_3t_2t_1t_1t_2t_3 \implies Mt_1t_2t_3t_1 = Mt_1t_2t_3 \in [123],$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ xy^2xy^2t_1t_2t_3t_1t_3t_2t_1 \in M)$$
$$t_1t_2t_3t_2 = xy^2t_1t_2t_3t_1t_3t_2t_1t_3t_2t_1 \implies Mt_1t_2t_3t_2 = Mt_3t_2t_1 \in [123],$$
$$(since\ \{N(t_1t_2t_3)^n|n \in N\}\ and\ xy^2t_1t_2t_3t_1t_3t_2t_1 \in M)$$
$$Mt_1t_2t_3t_3 = Mt_1t_2 \in [12]$$

$\boldsymbol{Mt_1t_2t_1t_3N}$

Continuing with the new double coset $Mt_1t_2t_1t_3N$, we find the coset stabilizer $N^{(1213)} = N^{1213} = \langle e \rangle$. But $Mt_1t_2t_1t_3$ is not distinct, since
$Mt_1t_2t_1t_3 = xy^{-1}xyt_3t_2t_1t_2t_3t_2t_1t_3t_2t_3t_1$ where $Mt_3t_2t_3t_1 \in Mt_1t_2t_1t_3$. Now
$M(t_1t_2t_1t_3)^{(1,3)} = Mt_3t_2t_3t_1$. Hence $(1,3) \in N^{(1213)}$. Therefore,

$$N^{(1213)} \geq \langle e, (1,3) \rangle.$$

The order of $N^{(1213)} = 2$. Thus the number of single cosets in $Mt_1t_2t_1t_3N$ is $\frac{|N|}{|N^{(1213)}|} = \frac{6}{2} = 3$. Since we know only 3 single cosets exist in the double coset [1213], now we find the orbits of $N^{(1213)}$ on $\{t_1, t_2, t_3\}$ are:

$$\mathcal{O} = \{2\}, \{1, 3\}.$$

Now take an element from each orbit and multiply on the right of the single coset representative $Mt_1t_2t_1t_3$ of the double coset $Mt_1t_2t_1t_3N$ to get the following:

$$Mt_1t_2t_1t_3t_2 \in [12132],$$
$$Mt_1t_2t_1t_3t_3 = Mt_1t_2t_1 \in [121]$$

We get a new double coset $Mt_1t_2t_1t_3t_2N$ with single coset representative $Mt_1t_2t_1t_3t_2$, denoted by [12132].

$Mt_1t_2t_1t_3t_2N$

Continuing with the new double coset $Mt_1t_2t_1t_3t_2N$, we find the coset stabilizer $N^{(12132)} = N^{12132} = \langle e \rangle$. But $Mt_1t_2t_1t_3t_2$ is not distinct, since $t_1t_2t_1t_3t_2 = xy^{-1}xyt_3t_2t_1t_2t_3t_2t_1t_3t_2t_3t_1t_2$ where $Mt_3t_2t_3t_1t_2 \in Mt_1t_2t_1t_3t_2$. Now $M(t_1t_2t_1t_3t_2)^{(1,3)}$ $= Mt_3t_2t_3t_1t_2$. Hence $(1,3) \in N^{(12132)}$. Therefore,

$$N^{(12132)} \geq \langle e, (1,3) \rangle.$$

The order of the coset stabiliser of $N^{(12132)} is 2$. Thus the number of single cosets in $Mt_1t_2t_1t_3t_2N$ is $\frac{|N|}{|N^{(12132)}|} = \frac{6}{2} = 3$. Since we know only 3 single cosets exist in [12132], now we find the orbits of $N^{(12132)}$ on $\{t_1, t_2, t_3\}$ which are:

$$\mathcal{O} = \{2\}, \{1,3\}.$$

Now take an element from each orbit and multiply on the right of the single coset representative $Mt_1t_2t_1t_3t_2$ of the double coset $Mt_1t_2t_1t_3t_2N$ to get the following:

$$Mt_1t_2t_1t_3t_2t_2 = Mt_1t_2t_1t_3 \in [1213],$$
$$t_1t_2t_1t_3t_2t_1 = xyxy^{-1}t_2t_3t_1t_3t_2t_3t_1t_2t_3t_2t_1t_3 \implies Mt_1t_2t_1t_3t_2t_1 = Mt_2t_3$$
$$t_2t_1t_3 \in [12132], (since\ \{N(t_1t_2t_1t_3t_2)^n | n \in N\}\ and\ xyxy^{-1}t_2t_3t_1t_3t_2t_3t_1 \in M)$$

We have completed the double coset enumeration, since the set of right cosets is closed under right multiplication then the index of M in G is 28 . We conclude:

$$G = MeN \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_2t_1N \cup Mt_1t_2t_3N \cup Mt_1t_2t_1t_3N \cup Mt_1t_2t_1t_3t_2N$$

where

$$G = \frac{2^{*3} : S_3}{(xt)^7, (xyt^x)^2, (yt)^9}$$

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(1213)}|} + \frac{|N|}{|N^{(12132)}|} \times |M|$$

$$|G| \leq (1+3+6+6+6+3+3) \times 18$$

$|G| \leq 28 \times 18$

$|G| \leq 504.$

The Cayley diagram summarizes the information listed above.

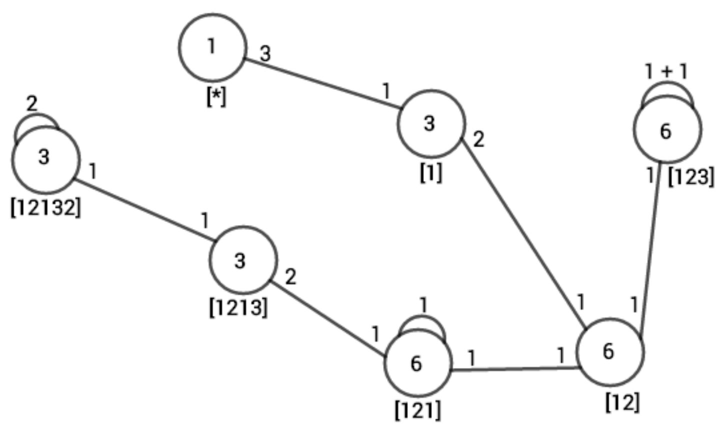

Figure 6.3: Cayley graph of $PSL(2,8)$ over $M = (9:2)$

Chapter 7

Monomial Representative

7.1 Lifting Linear Character Table of H

Definition 7.1. Kernel *of* $\chi = \{g \in G | \chi(g) = \chi(1)\}$.*[Rot95]*

Theorem 7.2. *Linear character of* H *are lifts of linear characters of* H/H'. *All irreducible characters of* H/H' *is abelian group implies to the number of irreducible of* H/H' *equals to the number of conjugacy classes of* $H/H' = |H|/|H'|$. *[Why06]*

In this section, we are going to induce a linear character of a proper subgroup up to G, to do so we need to investigate the subgroup H in group G. Let $G \cong 6^{\bullet}(5:2)$ with subgroup H, where $H \cong (3 \times 5) : 2$. Note, from a subgroup H there exist a normal subgroup, H'. Since H' is to be the derived group of H we have the generators of H', $H' =< (1, 14, 25, 7, 19)(2, 13, 26, 8, 20)(3, 16, 28, 9, 22)(4, 15, 27, 10, 21)$
$(5, 17, 29, 11, 24)(6, 18, 30, 12, 23) >\cong 5$.

Now we are going to find the character table of H/H'. Note $H/H' \cong 6$. Since H/H' is of order 6 then the character of $H/H' = \mathbb{Z}_2 \times \mathbb{Z}_3 = \{e, a, b, ab, ab^2, b^2\}$. Now that we know what H/H' is composed of, we can obtained the generators of H/H'. Since $H/H' =< H'a, H'b >\cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Thus the set is $\{H'e, H'a, H'b, H'ab, H'ab^2, H'b^2\}$. Note that $\mathbb{Z}_2 = \{0, 1\}$ with primitive square root of unity is -1 and $\mathbb{Z}_3 = \{0, 1, 2\}$ with primitive third root of unity is w.

Now we simplify each column and row. Note for the first row is always 1. Moving on, for the second row, χ_2, we get the following C_1 $((-1)^1)^0((w)^0)^0 =$

Table 7.1: Character Table of H/H’

$a \quad b$		(a^0b^0)	(a^1b^0)	(a^0b^1)	(a^0b^2)	(a^1b^1)	(a^1b^2)
	Classes	C_1	C_2	C_3	C_4	C_5	C_6
	Size	1	1	1	1	1	1
	Order	1	2	3	6	3	6
$(-1)^0(w)^0$	$\chi_{.1}$	$((-1)^0)^0((w)^0)^0$	$((-1)^0)^1((w)^0)^0$	$((-1)^0)^0((w)^0)^1$	$((-1)^0)^0((w)^0)^2$	$((-1)^0)^1((w)^0)^1$	$((-1)^0)^1((w)^0)^2$
$(-1)^1(w)^0$	$\chi_{.2}$	$((-1)^1)^0((w)^0)^0$	$((-1)^1)^1((w)^0)^0$	$((-1)^1)^0((w)^0)^1$	$((-1)^1)^0((w)^0)^2$	$((-1)^1)^1((w)^0)^1$	$((-1)^1)^1((w)^0)^2$
$(-1)^0(w)^1$	$\chi_{.3}$	$((-1)^0)^0((w)^1)^0$	$((-1)^0)^1((w)^1)^0$	$((-1)^0)^0((w)^1)^1$	$((-1)^0)^0((w)^1)^2$	$((-1)^0)^1((w)^1)^1$	$((-1)^0)^1((w)^1)^2$
$(-1)^1(w)^1$	$\chi_{.4}$	$((-1)^1)^0((w)^1)^0$	$((-1)^1)^1((w)^1)^0$	$((-1)^1)^0((w)^1)^1$	$((-1)^1)^0((w)^1)^2$	$((-1)^1)^1((w)^1)^1$	$((-1)^1)^1((w)^1)^2$
$(-1)^0(w)^2$	$\chi_{.5}$	$((-1)^0)^0((w)^2)^0$	$((-1)^0)^1((w)^2)^0$	$((-1)^0)^0((w)^2)^1$	$((-1)^0)^0((w)^2)^2$	$((-1)^0)^1((w)^2)^1$	$((-1)^0)^1((w)^2)^2$
$(-1)^1(w)^2$	$\chi_{.6}$	$((-1)^1)^0((w)^2)^0$	$((-1)^1)^1((w)^2)^0$	$((-1)^1)^0((w)^2)^1$	$((-1)^1)^0((w)^2)^2$	$((-1)^1)^1((w)^2)^1$	$((-1)^1)^1((w)^2)^2$

1, C_2 $((-1)^1)^1((w)^0)^0 = -1$, C_3 $((-1)^1)^0((w)^0)^1 = 1$, C_4 $((-1)^1)^0((w^0)^2 = 1$, C_5 $((-1)^1)^1((w)^0)^1 = -1$, and C_6 $((-1)^1)^1((w)^0)^2 = -1$. Also, since w is of order three we changed $w^4 = w$. Using the same procedure for the rest of χ and its classes, we completed table of H/H' where the conjugacy classes are listed below:

Let a=(1, 2)(3, 10)(4, 9)(5, 18)(6, 17)(7, 26)(8, 25)(11, 12)(13, 19)(14,20) (15, 28)(16, 27)(21, 22)(23, 29)(24, 30),

b= (1, 21, 11)(2, 22, 12)(3, 23, 13)(4, 24, 14)(5, 25, 15)(6, 26, 16)(7,27, 17)(8, 28, 18) (9, 30, 20)(10, 29, 19),

$a * b$= (1, 22, 11, 2, 21, 12)(3, 29, 13, 10, 23, 19)(4, 30, 14, 9, 24,20)(5, 8, 15, 18, 25, 28) (6, 7, 16, 17, 26, 27),

b^2 =(1, 11, 21)(2, 12, 22)(3, 13, 23)(4, 14, 24)(5, 15, 25)(6, 16, 26)(7,17, 27)(8, 18, 28) (9, 20, 30)(10, 19, 29),

$a * b^2$ = (1, 12, 21, 2, 11, 22)(3, 19, 23, 10, 13, 29)(4, 20, 24, 9, 14,30)(5, 28, 25, 18, 15, 8)(6, 27, 26, 17, 16, 7)

Table 7.2: Simplified Character Table of H/H’

Classes	C_1	C_2	C_3	C_4	C_5	C_6
Rep.	e	a	b	b^2	ab	ab^2
Size	1	1	1	1	1	1
χ_1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	1	-1	-1
$\chi_{.3}$	1	1	w	w^2	w	w^2
$\chi_{.4}$	1	-1	w	w^2	$-w$	$-w^2$
$\chi_{.5}$	1	1	w^2	$w^4 = w$	w^2	$w^4 = w$
$\chi_{.6}$	1	-1	w^2	$w^4 = w$	$-w^2$	$-w^4 = -w$

Since we have completed the linear character table of H/H', now we proceed by lifting from H/H' to H. By the definition of **lift**, $H' \lhd H$ where $\chi_i^\bullet$ is a character of H' and χ_i is a character of H then we have:

$$\chi_i(h) = \chi_i^\bullet(h),\ h\ \in H.$$

Before we lift from H/H' to H we determine the conjugacy classes of H. We are going to label every single conjugacy class in the following form listed below.

Let $h_1 = (1,2)(3,10)(4,9)(5,18)(6,17)(7,26)(8,25)(11,12)(13,19)(14,20)$
$(15,28)(16,27)(21,22)(23,29)(24,30)$,

$h_2 = (1,21,11)(2,22,12)(3,23,13)(4,24,14)(5,25,15)(6,26,16)(7,27,17)(8,28,18)(9,$
$30,20)(10,29,19)$,

$h_3 = (1,7,14,19,25)(2,8,13,20,26)(3,9,16,22,28)(4,10,15,21,27)(5,11,17,24,29)$
$(6,12,18,23,30)$,

$h_2^2 = (1,11,21)(2,12,22)(3,13,23)(4,14,24)(5,15,25)(6,16,26)(7,17,27)(8,18,28)$
$(9,20,30)(10,19,29)$,

$h_3^2 = (1,14,25,7,19)(2,13,26,8,20)(3,16,28,9,22)(4,15,27,10,21)(5,17,29,11,$
$24)(6,18,30,12,23)$, $h_1 * h_2 = (1,22,11,2,21,12)(3,29,13,10,23,19)(4,30,14,9,24,20)$
$(5,8,15,18,25,28)(6,7,16,17,26,27)$,

$h_1 * h_2^2 = (1,12,21,2,11,22)(3,19,23,10,13,29)(4,20,24,9,14,30)(5,28,25,18,15,$
$8)(6,27,26,17,16,7)$,

$h_2 * h_3 = (1,27,24,19,15,11,7,4,29,25,21,17,14,10,5)(2,28,23,20,16,12,8,3,30,$
$26,22,18,13,9,6)$,

$h_2^2 * h_3^2 = (1,24,15,7,29,21,14,5,27,19,11,4,25,17,10)(2,23,16,8,30,22,13,6,$
$28,20,12,3,26,18,9)$,

$h_3^2 * h_2 = (1,4,5,7,10,11,14,15,17,19,21,24,25,27,29)(2,3,6,8,9,12,13,16,18,20,$
$22,23,26,28,30)$, and

$h_2^2 * h_3 = (1,17,4,19,5,21,7,24,10,25,11,27,14,29,15)(2,18,3,20,6,22,8,23,9,26,$
$12,28,13,30,16)$.

Lift H/H' to H:

Note on the first row of $\chi_i(1)$ for each class $\chi_i^\bullet(1) = 1$. Thus, the value for the first row and column of $\chi_i(1)$ in the character table of H is 1. Moving on, to calculate the lift χ of a character $\chi^\bullet$ we have the following computations.

Find $\chi_2(h)$:

$\chi_2(h_1) = \chi_2^{\bullet}(H'h_1) = -1,$

$\chi_2(h_2) = \chi_2^{\bullet}(H'h_2) = 1,$

$\chi_2(h_2^2) = \chi_2^{\bullet}(H'h_2^2) = 1,$

$\chi_2(h_3) = \chi_2^{\bullet}(H'h_3) = 1,$ (since $h_3 \in e$)

$\chi_2(h_3^2) = \chi_2^{\bullet}(H'h_3^2) = 1,$ (since $h_3^2 \in e$)

$\chi_2(h_1h_2) = \chi_2^{\bullet}(H'h_1h_2) = -1,$

$\chi_2(h_1h_2^2) = \chi_2^{\bullet}(H'h_1h_2^2) = -1,$

$\chi_2(h_2h_3) = \chi_2^{\bullet}(H'h_2h_3) = 1,$ (since $h_2h_3 \in b$)

$\chi_2(h_2^2h_3^2) = \chi_2^{\bullet}(H'h_2^2h_3^2) = 1,$ (since $h_2^2h_3^2 \in b^2$)

$\chi_2(h_3^2h_2) = \chi_2^{\bullet}(H'h_3^2h_2) = 1,$ (since $h_3^2h_2 \in b$)

$\chi_2(h_2^2h_3) = \chi_2^{\bullet}(H'h_2^2h_3) = 1$ (since $h_2^2h_3 \in b^2$)

Find $\chi_3(h)$:

$\chi_3(h_1) = \chi_3^{\bullet}(H'h_1) = 1,$

$\chi_3(h_2) = \chi_3^{\bullet}(H'h_2) = w,$

$\chi_3(h_2^2) = \chi_3^{\bullet}(H'h_2^2) = w^2,$

$\chi_3(h_3) = \chi_3^{\bullet}(H'h_3) = 1,$ (since $h_3 \in e$)

$\chi_3(h_3^2) = \chi_3^{\bullet}(H'h_3^2) = 1,$ (since $h_3^2 \in e$)

$\chi_3(h_1h_2) = \chi_3^{\bullet}(H'h_1h_2) = w,$

$\chi_3(h_1h_2^2) = \chi_3^{\bullet}(H'h_1h_2^2) = w^2,$

$\chi_3(h_2h_3) = \chi_3^{\bullet}(H'h_2h_3) = w,$ (since $h_2h_3 \in b$)

$\chi_3(h_2^2h_3^2) = \chi_3^{\bullet}(H'h_2^2h_3^2) = w^2,$ (since $h_2^2h_3^2 \in b^2$)

$\chi_3(h_3^2h_2) = \chi_3^{\bullet}(H'h_3^2h_2) = w,$ (since $h_3^2h_2 \in b$)

$\chi_3(h_2^2h_3) = \chi_3^{\bullet}(H'h_2^2h_3) = w^2$ (since $h_2^2h_3 \in b^2$)

Find $\chi_4(h)$:

$\chi_4(h_1) = \chi_4^{\bullet}(H'h_1) = -1,$

$\chi_4(h_2) = \chi_4^{\bullet}(H'h_2) = w,$

$\chi_4(h_2^2) = \chi_4^{\bullet}(H'h_2^2) = w^2,$

$\chi_4(h_3) = \chi_4^{\bullet}(H'h_3) = 1,$ (since $h_3 \in e$)

$\chi_4(h_3^2) = \chi_4^{\bullet}(H'h_3^2) = 1,$ (since $h_3^2 \in e$)

$\chi_4(h_1h_2) = \chi_4^{\bullet}(H'h_1h_2) = -w,$

$\chi_4(h_1h_2^2) = \chi_4^{\bullet}(H'h_1h_2^2) = -w^2,$

$\chi_4(h_2h_3) = \chi_4^{\bullet}(H'h_2h_3) = w,$ (since $h_2h_3 \in b$)

$\chi_4(h_2^2h_3^2) = \chi_4^{\bullet}(H'h_2^2h_3^2) = w^2,$ (since $h_2^2h_3^2 \in b^2$)

$\chi_4(h_3^2h_2) = \chi_4^{\bullet}(H'h_3^2h_2) = w,$ (since $h_3^2h_2 \in b$)

$\chi_4(h_2^2h_3) = \chi_4^{\bullet}(H'h_2^2h_3) = w^2$ (since $h_2^2h_3 \in b^2$)

Find $\chi_5(h)$:

$\chi_5(h_1) = \chi_5^{\bullet}(H'h_1) = 1,$

$\chi_5(h_2) = \chi_5^{\bullet}(H'h_2) = w^2,$

$\chi_5(h_2^2) = \chi_5^{\bullet}(H'h_2^2) = w,$

$\chi_5(h_3) = \chi_5^{\bullet}(H'h_3) = 1,$

$\chi_5(h_3^2) = \chi_5^{\bullet}(H'h_3^2) = 1,$

$\chi_5(h_1h_2) = \chi_5^{\bullet}(H'h_1h_2) = w^2,$

$\chi_5(h_1h_2^2) = \chi_5^{\bullet}(H'h_1h_2^2) = w,$

$\chi_5(h_2h_3) = \chi_5^{\bullet}(H'h_2h_3) = w^2,$

$\chi_5(h_2^2h_3^2) = \chi_5^{\bullet}(H'h_2^2h_3^2) = w,$

$\chi_5(h_3^2h_2) = \chi_5^{\bullet}(H'h_3^2h_2) = w^2,$

$\chi_5(h_2^2h_3) = \chi_5^{\bullet}(H'h_2^2h_3) = w$

Find $\chi_6(h)$:

$\chi_6(h_1) = \chi_6^{\bullet}(H'h_1) = -1,$

$\chi_6(h_2) = \chi_6^{\bullet}(H'h_2) = w^2,$

$\chi_6(h_2^2) = \chi_6^{\bullet}(H'h_2^2) = w,$

$\chi_6(h_3) = \chi_6^{\bullet}(H'h_3) = 1,$

$\chi_6(h_3^2) = \chi_6^{\bullet}(H'h_3^2) = 1,$

$\chi_6(h_1h_2) = \chi_6^{\bullet}(H'h_1h_2) = -w^2,$

$\chi_6(h_1h_2^2) = \chi_6^{\bullet}(H'h_1h_2^2) = -w^4 = -w,$

$\chi_6(h_2h_3) = \chi_6^{\bullet}(H'h_2h_3) = w^2,$

$\chi_6(h_2^2h_3^2) = \chi_6^{\bullet}(H'h_2^2h_3^2) = w^4 = w,$

$\chi_6(h_3^2h_2) = \chi_6^{\bullet}(H'h_3^2h_2) = w^2,$

$\chi_6(h_2^2h_3) = \chi_6^{\bullet}(H'h_2^2h_3) = w^4 = w.$

Hence the lift of $\chi_i(h) = \chi_i^{\bullet}(h),\ h\ \in H$ and $2 < i < 6$. In other words we can lift all six linear characters of $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ to obtain $\chi_1, \chi_2, ..., \chi_6$ of G. The work is shown on the following tables.

Table 7.3: Lifted Character Table of H

Classes Rep. Size	C_1 e 1	C_2 h_1 5	C_3 h_2 1	C_4 h_2^2 1	C_5 h_3 2	C_6 h_3^2 2	C_7 h_1h_2 5	C_8 $h_1h_2^2$ 5
χ_1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	1	1	1	-1	-1
$\chi_{.3}$	1	1	w	w^2	1	1	w	w^2
$\chi_{.4}$	1	-1	w	w^2	1	1	$-w$	$-w^2$
$\chi_{.5}$	1	1	w^2	w	1	1	w^2	w
$\chi_{.6}$	1	-1	w^2	w	1	1	$-w^2$	$-w^4 = -w$

Table 7.4: Continue Lifted Character Table of H

Classes Rep. Size	C_9 h_2h_3 2	C_{10} $h_2^2h_3^2$ 2	C_{11} $h_3^2h_2$ 2	C_{12} $h_2^2h_3$ 2
χ_1	1	1	1	1
$\chi_{.2}$	1	1	1	1
$\chi_{.3}$	w	w^2	w	w^2
$\chi_{.4}$	w	w^2	w	w^2
$\chi_{.5}$	w^2	w	w^2	w
$\chi_{.6}$	w^2	$w^4 = w$	w^2	w

Therefore we have build the character table of H by using the lifting method. For the following sections of this chapter we used the same process, lifting, to build the character tables of H of each monomial progenitor.

7.1.1 Monomial Progenitor $53^{*2} :_m (13 : 4)$

To construct a monomial presentation of $53^{*2} :_m (13 : 4)$, we must induce a linear character from a subgroup H of G. We must choose a subgroup with index matching the degree of an irreducible character of G by considering the character table of G in Table 7.5. Note G has characters $\chi_{.1}, \chi_{.2}, ..., \chi_{.8}$. We proceed using $\chi_{.3}$ and look for a subgroup of order 2 so that $\frac{|G|}{|H|} = \frac{52}{26} = 2$.

Since the index of the two groups is 2 and the matrix representation is faithful then $A(xx)$ and $A(yy)$ will be represented by a 2×2 matrices.

Verifying the Induction

We produce a character table for $\chi_{.3}$ in Table 7.5. First we will verify the induction $\chi_{.4}$ of H to $\chi_{.3}$ of G by considering the irreducible characters ϕ of H and ϕ^G of G. G is generated by x and y, where $x = (1,2,3,4,5,6,7,8,9,10,11,12,13)$ and $y = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$.

The Conjugacy classes of group G are
$C1 = Id(G)$
$C2 = (1,3)(4,13)(5,12)(6,11)(7,10)(8,9), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7), \ldots$ $C3 = (1,2,3,4,5,6,7,8,9,10,11,12,13), (1,13,12,11,10,9,8,7,6,5,4,3,2)$
$C4 = (1,3,5,7,9,11,13,2,4,6,8,10,12), (1,12,10,8,6,4,2,13,11,9,7,5,3)$.
$C5 = (1,11,8,5,2,12,9,6,3,13,10,7,4), (1,4,7,10,13,3,6,9,12,2,5,8,11)$
$C6 = (1,10,6,2,11,7,3,12,8,4,13,9,5), (1,5,9,13,4,8,12,3,7,11,2,6,10)$
$C7 = (1,9,4,12,7,2,10,5,13,8,3,11,6), (1,6,11,3,8,13,5,10,2,7,12,4,9)$
$C8 = (1,8,2,9,3,10,4,11,5,12,6,13,7), (1,7,13,6,12,5,11,4,10,3,9,2,8)$

Consider the subgroup H of G given below.
$H = Id(G), (1,12,10,8,6,4,2,13,11,9,7,5,3)$

The conjugacy classes of H are
$D1 = Id(G)$
$D2 = (1,12,10,8,6,4,2,13,11,9,7,5,3)$
$D3 = (1,10,6,2,11,7,3,12,8,4,13,9,5)$
$D4 = (1,8,2,9,3,10,4,11,5,12,6,13,7)$
$D5 = (1,6,11,3,8,13,5,10,2,7,12,4,9)$
$D6 = (1,4,7,10,13,3,6,9,12,2,5,8,11)$
$D7 = (1,2,3,4,5,6,7,8,9,10,11,12,13)$
$D8 = (1,13,12,11,10,9,8,7,6,5,4,3,2)$

$D9 = (1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$
$D10 = (1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3, 11, 6)$
$D11 = (1, 7, 13, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8)$
$D12 = (1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)$
$D13 = (1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)$

From the character tables of G and H we are going to use the information we labeled as ϕ of H and ϕ^G of G. By the definition of induction we induce the character $\phi = \chi.4$ of H up to $\phi^G = \chi.3$ of G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{52}{26} = 2$.

$\phi_1^G = \frac{2}{1} \sum_{w \in H \cap C_1} \phi(w)$

which implies $\phi_1^G = \frac{2}{1}(\phi(1)) = 2(1) = 2$.

$\phi_2^G = \frac{2}{13} \sum_{w \in H \cap C_2} \phi(w)$

$\implies \phi_2^G = \frac{2}{13} \sum_{w \in H \cap C_2} \phi(w)$=0 (since $H \cap C_2 = \phi$)

$\phi_3^G = \frac{2}{2} \sum_{w \in H \cap C_3} \phi(w)$

$\implies \phi_3^G = 1(1\phi((1, 2, 3, 4, 5, ..., 13) + 1\phi((1, 13, 12, 11, ...)) = Z1\#5 + Z1\#8$

$\phi_4^G = \frac{2}{2} \sum_{w \in H \cap C_4} \phi(w)$

$\implies \phi_4^G = 1(1\phi((1, 3, 5, 7, 9...) + 1\phi((1, 12, 10, 8, ...)) = Z1\#10 + Z1\#3$
$\phi_5^G = \frac{2}{2} \sum_{w \in H \cap C_5} \phi(w)$

$\implies \phi_5^G = 1(1\phi((1, 11, 8, 5, 2...) + 1\phi((1, 4, 7, 10, 13...)) = Z1\#11 + Z1\#2$

$\phi_6^G = \frac{2}{2}\sum_{w\in H\cap C_6}\phi(w)$

$\implies \phi_6^G = 1(1\phi((1,10,6,2,11...)+1\phi((1,5,9,13,4...)) = Z1\#6 + Z1\#7$

$\phi_7^G = \frac{2}{2}\sum_{w\in H\cap C_7}\phi(w)$

$\implies \phi_7^G = 1(1\phi((1,9,4,12,...)+1\phi((1,6,11,3,8,...)) = Z1 + Z1\#12$

$\phi_8^G = \frac{2}{2}\sum_{w\in H\cap C_8}\phi(w)$

$\implies \phi_8^G = 1(1\phi((1,8,2,9,3,...)+1\phi((1,7,13,6,...)) = Z1\#9 + Z1\#4$

So $\phi\uparrow_H^G = (2, 0, Z1\#8+Z1\#5, Z1\#10+Z1\#3, Z1\#11+Z1\#2, Z1\#7+Z1\#6, Z1\#12+ Z1, Z1\#9 + Z1\#4)$ and we have verified that $\chi_{.4}$ of H induces $\chi_{.3}$ of G.

Instead of writing $zeta(13)_{13}^r$ for $r \in 1 \leq r \leq 12$, we are going to replace it with $Z1\#r$ for r. So we have verified $\phi\uparrow_H^G$ is equivalent to $\chi_{.3}$

```
> CG[3];
( 2, 0, zeta(13)_13^8 + zeta(13)_13^5, zeta(13)_13^10
+ zeta(13)_13^3, zeta(13)_13^11 + zeta(13)_13^2,
zeta(13)_13^7 + zeta(13)_13^6,-zeta(13)_13^11
- zeta(13)_13^10- zeta(13)_13^9 - zeta(13)_13^8
 - zeta(13)_13^7 - zeta(13)_13^6 - zeta(13)_13^5
 - zeta(13)_13^4 - zeta(13)_13^3 - zeta(13)_13^2 - 1,
zeta(13)_13^9 + zeta(13)_13^4 )
```

Note, since we are in Root Unity 13 we have the following:

$$Z1^{13} = 1,$$

$$Z1^{13} - 1 = 0,$$

$$(Z1-1)(Z1^{12}+Z1^{11}+Z1^{10}+Z1^9+Z1^8+Z1^7+Z1^6+Z1^5+Z1^4+Z1^3Z1^2+Z1+1) = 0.$$

Using the work from above we changed $-zeta's$ to $(Z1\#12 + Z1)$ to rewrite $CG[3]$ in terms of $Z1\#r$

Thus,

$CG[3] = (2, 0, Z1\#8 + Z1\#5, Z1\#10 + Z1\#3, Z1\#11 + Z1\#2, Z1\#7 + Z1\#6, Z1\#12 + Z1, Z1\#9 + Z1\#4)$.

Now we print the Characters tables of G and H.

Table 7.5: Character Table of G

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
$\chi_{.1}$	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	1	1	1	1	1
$\chi_{.3}$	2	0	Z1	$Z1\#2$	$Z1\#3$	$Z1\#4$	$Z1\#5$	$Z1\#6$
$\chi_{.4}$	2	0	$Z1\#5$	$Z1\#3$	$Z1\#2$	$Z1\#6$	Z1	$Z1\#4$
$\chi_{.5}$	2	0	$Z1\#2$	$Z1\#4$	$Z1\#6$	$Z1\#5$	$Z1\#3$	Z1
$\chi_{.6}$	2	0	$Z1\#4$	$Z1\#5$	$Z1$	$Z1\#3$	$Z1\#6$	$Z1\#2$
$\chi_{.7}$	2	0	Z1#6	Z1	$Z1\#5$	$Z1\#2$	$Z1\#4$	$Z1\#3$
$\chi_{.8}$	2	0	Z1#3	Z1#6	$Z1\#4$	$Z1$	$Z1\#2$	$Z1\#5$

# denotes algebraic conjugation.
$Z1$ is the primitive thirteen root of unity.

Table 7.6: Character Table of H

χ	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
$\chi_{.1}$	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	Z1	$Z1\#2$	$Z1\#3$	$Z1\#4$	$Z1\#5$	$Z1\#6$	$Z1\#7$
$\chi_{.3}$	1	$Z1\#2$	$Z1\#4$	$Z1\#6$	$Z1\#8$	$Z1\#10$	$Z1\#12$	Z1
$\chi_{.4}$	1	$Z1\#3$	$Z1\#6$	$Z1\#9$	$Z1\#12$	$Z1\#2$	$Z1\#5$	$Z1\#8$
$\chi_{.5}$	1	$Z1\#4$	$Z1\#8$	$Z1\#12$	$Z1\#3$	$Z1\#7$	$Z1\#11$	$Z1\#2$
$\chi_{.6}$	1	$Z1\#5$	$Z1\#10$	$Z1\#2$	$Z1\#7$	$Z1\#12$	$Z1\#4$	$Z1\#9$
$\chi_{.7}$	1	$Z1\#6$	$Z1\#12$	$Z1\#5$	$Z1\#11$	$Z1\#4$	$Z1\#10$	$Z1\#3$
$\chi_{.8}$	1	$Z1\#7$	Z1	$Z1\#8$	$Z1\#2$	$Z1\#9$	$Z1\#3$	$Z1\#10$
$\chi_{.9}$	1	$Z1\#8$	$Z1\#3$	$Z1\#11$	$Z1\#6$	Z1	$Z1\#9$	$Z1\#4$
$\chi_{.10}$	1	$Z1\#9$	$Z1\#5$	Z1	$Z1\#10$	$Z1\#6$	$Z1\#2$	$Z1\#11$
$\chi_{.11}$	1	$Z1\#10$	$Z1\#7$	$Z1\#4$	Z1	$Z1\#11$	$Z1\#8$	$Z1\#5$
$\chi_{.12}$	1	$Z1\#11$	$Z1\#9$	$Z1\#7$	$Z1\#5$	$Z1\#3$	Z1	$Z1\#12$
$\chi_{.13}$	1	$Z1\#12$	$Z1\#11$	$Z1\#10$	$Z1\#9$	$Z1\#8$	$Z1\#7$	$Z1\#6$

Table 7.7: Character Table of H Cont.

χ	D_9	D_{10}	D_{11}	D_{12}	D_{13}
$\chi.1$	1	1	1	1	1
$\chi.2$	$Z1\#8$	$Z1\#9$	$Z1\#10$	$Z1\#11$	$Z1\#12$
$\chi.3$	$Z1\#3$	$Z1\#5$	$Z1\#7$	$Z1\#9$	$Z1\#11$
$\chi.4$	$Z1\#11$	$Z1$	$Z1\#4$	$Z1\#7$	$Z1\#10$
$\chi.5$	$Z1\#6$	$Z1\#10$	Z1	$Z1\#5$	$Z1\#9$
$\chi.6$	Z1	$Z1\#6$	$Z1\#11$	$Z1\#3$	$Z1\#8$
$\chi.7$	$Z1\#9$	$Z1\#2$	$Z1\#8$	Z1	$Z1\#7$
$\chi.8$	$Z1\#4$	$Z1\#11$	$Z1\#5$	$Z1\#12$	$Z1\#6$
$\chi.9$	$Z1\#12$	$Z1\#7$	$Z1\#2$	$Z1\#10$	$Z1\#5$
$\chi.10$	$Z1\#7$	$Z1\#3$	$Z1\#12$	$Z1\#8$	$Z1\#4$
$\chi.11$	$Z1\#2$	$Z1\#12$	$Z1\#9$	$Z1\#6$	$Z1\#3$
$\chi.12$	$Z1\#10$	$Z1\#8$	$Z1\#6$	$Z1\#4$	$Z1\#2$
$\chi.13$	$Z1\#5$	$Z1\#4$	$Z1\#3$	$Z1\#2$	Z1

# denotes algebraic conjugation.
Z_1 is the primitive thirteen root of unity.

Before we verify the monomial representation of the matrices we must find the exact value of $Z1$. Noticed the order $| < Z > | = 13$, that means we must find the GFq(field) $\ni \frac{13}{p-1}$ has a subgroup isomorphic to $< Z >$ prime $p \ \ni \ \frac{13}{p-1}$. Then the prime is 53. Since $\frac{13}{53-1} = \frac{13}{52}$. Thus $Z_{53} - \{0\} =< 2 >$, since $2^{52} = 1 \ mod \ 53$. We apply the following formula: $|a^k| = \frac{|a|}{gcd(k,|a|)}$ where $a = 2$. Since the generator of the cyclotomic field 53 is 2 and $|2| = 52$.

Thus,

$$|2^k| = \frac{|2|}{gcd(k,|2|)}$$

$$\Rightarrow |2^k| = \frac{52}{gcd(k,52)}$$

where $|2^k| = 13$.

Hence, $k = 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48$.

$$|16| = |2^4| = 13$$

$$|44| = |2^8| = 13$$

$$|15| = |2^{12}| = 13$$

Take the first order $Z = 16$ where $k^a = 4^2 = 16$ mod 53. Now we can replace Z1 with 16 to calculate every entry of $CG[3] = \chi_{.3}, CH[4] = \chi_{.4}$, and the matrix.

Table 7.8: $\chi_{.3}$ of G

ϕ^G	Class	Size	Class Representative
2	C_1	1	Id(G)
0	C_2	13	(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
$Z1$	C_3	2	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
$Z1\#2$	C_4	2	(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)
$Z1\#3$	C_5	2	(1, 4, 7, 10, 13, 3, 6, 9, 12, 2, 5, 8, 11)
$Z1\#4$	C_6	2	(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)
$Z1\#5$	C_7	2	(1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9)
$Z1\#6$	C_8	2	(1, 7, 13, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8)

Table 7.9: $\chi_{.4}$ of H

ϕ	Class	Size	Class Representative
1	D_1	1	Id(H)
$Z1\#3 = 15$	D_2	1	(1, 12, 10, 8, 6, 4, 2, 13, 11, 9, 7, 5, 3)
$Z1\#6 = 13$	D_3	1	(1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5)
$Z1\#9 = 36$	D_4	1	(1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7)
$Z1\#12 = 10$	D_5	1	(1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9)
$Z1\#2 = 44$	D_6	1	(1, 4, 7, 10, 13, 3, 6, 9, 12, 2, 5, 8, 11)
$Z1\#5 = 24$	D_7	1	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
$Z1\#8 = 42$	D_8	1	(1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)
$Z1\#11 = 47$	D_9	1	(1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)
$Z1 = 16$	D_{10}	1	(1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3, 11, 6)
$Z1\#4 = 28$	D_{11}	1	(1, 7, 13, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8)
$Z1\#7 = 49$	D_{12}	1	(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)
$Z1\#10 = 46$	D_{13}	1	(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)

Equivalencies from table to construct matrix: $1 = 1$,
$Z_1 = 16$.

Now prove the monomial representation has the following generators:
$A(xx) = \begin{bmatrix} Z1^5 = 24 & 0 \\ 0 & Z1^8 = 42 \end{bmatrix}$, and $A(yy) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Verifying the Monomial Representation

Since we have a linear character ϕ of the subgroup H of index 2 in G we let $G = Ht_1 \cup Ht_2$ where the $t_i's$ are transversals of G acting on H.

That is $G = He \cup H(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$.

Continuing the process in a 2x2 matrix:

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}). \end{bmatrix}$$

We will calculate all 2 elements of the matrix using the following calculation where $t_1 = e$, $t_2 = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$,
and $x = (1,2,3,4,5,6,7,8,9,10,11,12,13)$ with each t_i corresponding to each transversal respectively from the G listed before. We will start by calculating the multiplication of t_ixt_j^{-1} and list the resulting permutation rather than show the entire process. For example:
$\phi(t_1xt_1^{-1}) = \phi(e(1,2,3,4,5,6,7,8,9,10,11,12,13)e^-1) = \phi((1,2,3,4,5,6,7,8,$
$9,10,11,12,13)$. We will go directly to what $\phi(t_1xt_1^{-1})$ is equal to. Since $(1,2,3,4,$
$5,6,7,8,9,10,11,12,13) \in H$. We look back to our Table 5.4 noticed
$\phi((1,2,3,4,5,6,7,8,9,10,11,12,13))$ is in class D_7=Z_1#5 $= 16^5 = 24$ *mod* 53 where $Z_1 = 16$. Therefore, the nonzero entry for *Row* 1 is 24. We proceed as follows:
Row1 :
$\phi(t_1xt_1^{-1}) = \phi((1,2,3,4,5,6,7,8,9,10,11,12,13)) = 24$
$\phi(t_1xt_2^{-1}) = \phi((1,11)(2,10)(3,9)(4,8)(5,7)(12,13)) = 0$
Row2 :
$\phi(t_2xt_1^{-1}) = \phi((1,2,3,4,5,6,7,8,9,10,11,12,13)) = 0$
$\phi(t_2xt_2^{-1}) = \phi((1,13,12,11,10,9,8,7,6,5,4,3,2)) = 42$

(since $\phi((1,13,12,11,10,9,8,7,6,5,4,3,2)) \in D_8$)

Each ϕ of H corresponded with a conjugacy class of either H or G. If the element is in a conjugacy class from H (seen in $Table$ 5.4) we write the value of ϕ for that class. Since our matrix was produced in cyclotomic field 53, we needed to produce an order 13 element in Z_{53}. In this case, 2 was chosen as the element of order 13. To complete this process, the matrix for yy should also be verified by repeating the process above where $y = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$.

Row1 :

$\phi(t_1 y t_1^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$

$\phi(t_1 y t_2^{-1}) = \phi(e) = 1$

Row2 :

$\phi(t_2 y t_1^{-1}) = \phi(e) = 0$

$\phi(t_2 y t_2^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$

Therefore,

$$A(xx) = \begin{bmatrix} 24 & 0 \\ 0 & 42 \end{bmatrix}, \text{ and } A(yy) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To determine if we have a faithful representation we check if the order of each matrix equals to the order of our generators. Since $|A(xx)| = 13$ and $|A(yy)| = 2$ with $|A(xx) \cdot A(yy)| = 2$ (the order of our index), we know that $\langle A(xx)$ and $A(yy)\rangle$ is a faithful representation of G. We proceed and produce a permutation representation.

Constructing a Permutation Representation

We worked in Z_{13}(number field 53) and on 2×2 matrices(dimension 2), thus we produce a $53^{*2} :_m (13:4)$ progenitor permutation representation based on the monomial representation $(13:4) = \langle A(xx), A(yy)\rangle$. Consider the matrix entries for $a_{i,j}$ for A(xx) and A(yy). We know there are 2 t's, one for each column of our matrix and because we have a semi-direct product in our progenitor then the elements of $(13:4)$ act as automorphisms of $< t_1 > * < t_2 >$.

Now we find the permutations for our two matrices xx and yy, to do so we use the following formula

$$a_{i,j} = 1 \text{ if the automorphism takes } t_i \to t_j$$

$$a_{i,j} = n \text{ if the automorphism takes } t_i \to t_j^n.$$

Since 53^{*2} is a free product of two cyclic groups of order 52 we will construct a table with two t's of order 52 labeled from 1...104 found in table 5.5. We are going to use each non-zero entry for each matrix $A(xx)$ and $A(yy)$ to apply the formula listed above.

$$a_{11} = 24 \textit{ and } a_{22} = 42 \;\;\&\;\; b_{12} = 1 \textit{ and } b_{21} = 1$$

Thus,

$$\begin{array}{cc} A(xx) & A(yy) \\ t_1 \to t_1^{24}, & t_1 \to t_2, \\ t_2 \to t_2^{42} & t_2 \to t_1 \end{array}$$

Proceeding on finding a permutation representative we must describe Table 7.10. The top number of the chart labels each element, the middle of the chart shows what the automorphism produces, and the bottom number shows what that produced element is numbered on the top of the chart. The same process is conducted for generators yy, but for Table 7.11 we will only show part of the process.

Table 7.10: Automorphisms of A(xx)

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{24}	t_2^{42}	t_1^{48}	t_2^{31}	t_1^{19}	t_2^{20}	t_1^{43}	t_2^9	t_1^{14}	t_2^{51}	t_1^{38}	t_2^{40}
47	83	95	61	37	40	85	18	27	102	75	80

13	14	15	16	17	18	19	20	21	22	23	24
t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^9	t_2^{29}	t_1^{33}	t_2^{18}	t_1^4	t_2^7	t_1^{28}	t_2^{49}	t_1^{52}	t_2^{38}	t_1^{23}	t_2^{27}
17	58	65	36	7	14	55	98	103	76	45	54

25	26	27	28	29	30	31	32	33	34	35	36
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{47}	t_2^{16}	t_1^{18}	t_2^5	t_1^{42}	t_2^{47}	t_1^{13}	t_2^{36}	t_1^{37}	t_2^{25}	t_1^8	t_2^{14}
93	32	35	10	83	94	25	72	73	50	15	28

37	38	39	40	41	42	43	44	45	46	47	48
t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{32}	t_2^3	t_1^3	t_2^{45}	t_1^{27}	t_2^{34}	t_1^{51}	t_2^{23}	t_1^{22}	t_2^{12}	t_1^{46}	t_2
63	6	5	90	53	68	101	46	43	24	91	2

49	50	51	52	53	54	55	56	57	58	59	60
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{17}	t_2^{43}	t_1^{41}	t_2^{32}	t_1^{12}	t_2^{21}	t_1^{36}	t_2^{10}	t_1^7	t_2^{52}	t_1^{31}	t_2^{41}
33	86	81	64	23	42	71	20	13	104	61	82

61	62	63	64	65	66	67	68	69	70	71	72
t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^2	t_2^{30}	t_1^{26}	t_2^{19}	t_1^{50}	t_2^8	t_1^{21}	t_2^{50}	t_1^{45}	t_2^{39}	t_1^{16}	t_2^{28}
3	60	51	38	99	16	41	100	89	78	31	56

73	74	75	76	77	78	79	80	81	82	83	84
t_1^{37}	t_2^{37}	t_1^{38}	t_2^{38}	t_1^{39}	t_2^{39}	t_1^{40}	t_2^{40}	t_1^{41}	t_2^{41}	t_1^{43}	t_2^{42}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{40}	t_2^{17}	t_1^{11}	t_2^{6}	t_1^{35}	t_2^{48}	t_1^{6}	t_2^{37}	t_1^{30}	t_2^{26}	t_1	t_2^{15}
79	34	21	12	69	96	11	74	59	52	1	30

85	86	87	88	89	90	91	92	93	94	95	96
t_1^{43}	t_2^{43}	t_1^{44}	t_2^{44}	t_1^{45}	t_2^{45}	t_1^{46}	t_2^{46}	t_1^{47}	t_2^{47}	t_1^{48}	t_2^{48}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{25}	t_2^{4}	t_1^{49}	t_2^{46}	t_1^{20}	t_2^{35}	t_1^{44}	t_2^{24}	t_1^{15}	t_2^{13}	t_1^{39}	t_2^{2}
49	8	97	92	39	70	87	48	29	26	77	4

97	98	99	100	101	102	103	104
t_1^{49}	t_2^{49}	t_1^{50}	t_2^{50}	t_1^{51}	t_2^{51}	t_1^{52}	t_2^{52}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{10}	t_2^{44}	t_1^{34}	t_2^{33}	t_1^{5}	t_2^{22}	t_1^{29}	t_2^{11}
19	88	67	66	9	44	57	22

Table 7.11: Automorphisms of A(yy)

1	2	3	4	...	101	102	103	104
t_1	t_2	t_3	t_4	...	t_1^{51}	t_2^{51}	t_1^{52}	t_2^{52}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_1	t_4	t_3	...	t_2^{51}	t_1^{51}	t_2^{52}	t_1^{52}
2	1	4	3	...	102	101	104	103

From Table 7.10 and 7.11, we can construct permutations for $A(xx)$ and $A(yy)$ by using our labels from each automorphism. Consider element #1 be denoted as t_1 from Table 5.5. This produces the permutation $(1, 47, 91, 87, 97, 19, 55, 71, 31,$
$25, 93, 29, 83)$. If we follow each element and its corresponding automorphism number labeling and repeating the process for xx, we produce the following permutations:

$A(xx) = (1, 47, 91, 87, 97, 19, 55, 71, 31, 25, 93, 29, 83),$
$(3, 95, 77, 69, 89, 39, 5, 37, 63, 51, 81, 59, 61),$
$(7, 85, 49, 33, 73, 79, 11, 75, 21, 103, 57, 13, 17),$
$(9, 27, 35, 15, 65, 99, 67, 41, 53, 23, 45, 43, 101),$
$(2, 84, 30, 94, 26, 32, 72, 56, 20, 98, 88, 92, 48),$
$(4, 62, 60, 82, 52, 64, 38, 6, 40, 90, 70, 78, 96),$
$(8, 18, 14, 58, 104, 22, 76, 12, 80, 74, 34, 50, 86),$
$(10, 102, 44, 46, 24, 54, 42, 68, 100, 66, 16, 36, 28)$

Likewise for yy.

$A(yy) = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16),$
$(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32),$
$(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48),$
$(49, 50)(51, 52)(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64),$
$(65, 66)(67, 68)(69, 70)(71, 72)(73, 74)(75, 76)(77, 78)(79, 80),$
$(81, 82)(83, 84)(85, 86)(87, 88)(89, 90)(91, 92)(93, 94)(95, 96),$
$(97, 98)(99, 100)(101, 102)(103, 104).$
Therefore, we have constructed a permutation representation from our matrices.

Creating a Presentation of the Progenitor

To construct a presentation for the progenitor we must choose a t to normalize from our two choices $< t_1 > * < t_2 >$. Let $t \sim t_1$. Now we find permutations which normalizes $< t_1 >$ or fixes the following set
$\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, t_1^{13}, t_1^{14}, ..., t_1^{52}\}$. This is a defining characteristic of a monomial progenitors. Monomial progenitors fix a set of $t's$ while permutation

progenitors fix only one specific t_i.

By using Magma we were able to find the stabiliser of t.

```
> Sch:=SchreierSystem(G,sub<G|Id(G)>);
>  ArrayP:=[Id(N): i in [1..#N]];
>  for i in [2..#N] do
for>  P:=[Id(N): l in [1..#Sch[i]]];
for>  for j in [1..#Sch[i]] do
for|for>  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for>  if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
for|for>  if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
for|for>  end for;
for>  PP:=Id(N);
for>  for k in [1..#P] do
for|for>  PP:=PP*P[k]; end for;
for>  ArrayP[i]:=PP;
for>  end for;
> Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,17,19,21,23,
>25,27,29,31,33,35,37,39,41,43,45,47,49,51,53,55,57,59,
>61,63,65,67,69,71,73,75,77,79,81,83,85,87,89,91,93, 95,
>97,99,101,103});

> Normaliser.1;

>A1:=N!(1, 55, 83, 19, 29, 97, 93, 87, 25, 91, 31, 47, 71)
>(2, 72, 48, 32, 92, 26, 88, 94, 98, 30, 20, 84, 56)
>(3, 5, 61, 39, 59, 89, 81, 69, 51, 77, 63, 95, 37)
>(4,38, 96, 64, 78, 52, 70, 82, 90, 60, 40, 62, 6)
>(7, 11, 17, 79, 13, 73, 57,33, 103, 49, 21, 85, 75)
>(8, 76, 86, 22, 50, 104, 34, 58, 74, 14, 80, 18,12)
>(9, 67, 101, 99, 43, 65, 45, 15, 23, 35, 53, 27, 41)
>(10, 42, 28, 54, 36,24, 16, 46, 66, 44, 100, 102, 68);

> for i in [1..26] do if ArrayP[i] eq A1 then Sch[i]; end if;
>end for;
x^6
```

To include permutation that fixes one to our presentation we must convert this permutation into words. The code from above was used to find the following: $\mathrm{A1} = x^6$. We look at the permutation $xx^6 = (1, 55, 83,)$, 1 goes to 55, so we check the label in our Table 7.10 where the element is 28, thus the automorphism is t_1^{28}

$$\text{A1} \rightarrow t^{x^6} = t_1^{28}.$$

So, a presentation for the progenitor is $53^{*2} :_m (13:4) = G < x, y, t >:= Group < x, y, t | y^2, (x^{-1} * y)^2, x^{(-13)}, t^{53}, t^{x^6} = t^{28} >$;

To check if our progenitor is correct we apply $Grindstaff's\ Lemma$. Our symmetric generators are t_1 and t_2. We want to add to the above presentation that all $ti's$ commute; that is, (t_1, t_2). Now $t_2 = t^y$.

So we check if

$53^{*4} :_m (13:4) = G < x, y, t >:= Group < x, y, t | y^2, (x^{-1} * y)^2, x^{(-13)}, t^{53}, t^{x^6} = t^{28} >$ factored by (t, t^y) is the group $53^2 :_m (13:4)$ of order $53^2 \times (13 \times 4)$.

```
 G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2, x^(-13),
 t^53,
 t^(x^6)=t^28,  (t,t^y)>;
print Index(G,sub<G|x,y>: CosetLimit:=9^10,
Hard:=true, Print:=2);
```

Next, we find finite homomorphic images of the progenitor $53^{*2} :_m (13:4)$. Thus, we factor the progenitor by additional relations. Here, we have factored by first order relations.

```
for r,s,u,v,w,z,aa,bb,cc,dd,ee,ff,gg,hh,ii,jj,
kk,ll,mm,nn,oo,pp,qq,rr in [0..10] do
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2, x^(-13),
t^53,
t^(x^6)=t^28,(y*(t^5)^(x^2))^r, (y*(t^3)^(x * y))^s,
(y*(t^3)^(y * x))^u, (y*(t^12)^(x^-2))^v,(y*(t^5)^(x^-2))^w,
(y*(t^12)^(x*y))^z, (y*t^(x * y^-1))^aa,  (y*(t^4)^(x^2))^bb,
(y*(t^3)^(x^3))^cc, (y*(t^12)^(x^-1))^dd, (y*(t^10)^(x))^ee,
(y*(t^7)^(x^-1))^ff,(y*(t^2)^(x^-2))^gg, (y*(t^2)^(x^-1))^hh,
(y*(t^3)^(x^2 * y))^ii, (y*(t^8)^(x * y))^jj,(y*(t^34)^(y))^kk,
(y*(t^2)^(x^3))^ll, (y*(t^10)^(x^2))^mm,(y*(t^6)^(x^-2))^nn,
(y*(t^6)^(y^-1 * x))^oo,(y*t^(x^2 * y^-1))^pp,
(y*(t^6)^(x^-1))^qq, (y*(t^41)^(y^2))^rr>;
if #G gt 26 then
#G,r,s,u,v,w,z,aa,bb,cc,dd,ee,ff,gg,hh,ii,jj,
kk,ll,mm,nn,oo,pp,qq,rr;
end if;
end for;
```

Homomorphic Images of $53^{*4} :_m (13:4)$ were found, please see Chapter 10.

Chapter 8

Progenitors with no Images Due to MAGMA Resources

8.1 Monomial Progenitor $7^{*8} :_m (3^2 : 8)$

To construct a monomial presentation of $7^{*2} :_m (3^2 : 8)$, we must induce a linear character from a subgroup H of G. We must choose a subgroup with index matching the degree of an irreducible character of G by considering the character table of G in Table 5.1 and note G has characters $\chi_{.1}, \chi_{.2}, ..., \chi_{.9}$. We proceed using $\chi_{.9}$ and look for a subgroup of order 8 so that $\frac{|G|}{|H|} = \frac{72}{9} = 8$.
Since the index of the two groups is 8 this implies, if a matrix representation is faithful then $A(xx), A(yy)$, and $A(zz)$ will be represented by a 8×8 matrices.

Verifying the Induction

We produce a character table for $\chi_{.9}$ in table 5.2. First we will verify the induction $\chi_{.2}$ of H to $\chi_{.9}$ of G by considering the irreducible characters ϕ of H and ϕ^G of G. G is generated by x, y, and z, where $x = (1,2,9)(3,4,5)(6,7,8)$, $y := (1,4,7)(2,5,8)(3,6,9)$, and $z = (1,6,4,5,2,3,8,7)$. But one of the generators is redundant, thus using 2 instead of 3 generators is efficient. Then the matrix representation will be $A(xx)$ and $A(zz)$.

The Conjugacy classes of group G are

$C1 = Id(G)$

$C2 = (1,4)(2,3)(5,9)(6,8),(1,7)(2,6)(3,5)(8,9),(1,6)(2,8)(3,4)(7,9),...$

$C3 = (1,2,9)(3,4,5)(6,7,8),(1,3,8)(2,4,6)(5,7,9),(1,9,2)(3,5,4)(6,8,7),...$

$C4 = (1,4,2,8)(3,7,6,5),(1,2,3,5)(4,6,9,7),(1,3,6,4)(2,7,8,9),...$

$C5 = (1,7,5,8)(2,3,4,9),(1,5,3,2)(4,7,9,6),(2,6,9,5)(3,7,8,4),..$

$C6 = (1,6,4,5,2,3,8,7),(1,8,6,3,7,9,2,5),(1,5,7,6,9,8,3,4),...$

$C7 = (1,9,7,2,5,3,8,4),(1,2,4,9,6,8,3,7),(1,5,8,6,2,7,4,3),...$

$C8 = (1,5,9,2,8,4,6,7),(1,6,2,9,3,7,5,4),(1,4,8,3,5,2,7,9),...$

$C9 = (1,2,6,5,8,7,9,4),(2,8,6,4,9,3,5,7),(1,9,5,6,3,4,2,7),...$

Consider the subgroup H of G given below.

$H = Id(G),(1,6,5)(2,7,3)(4,9,8),(1,3,8)(2,4,6)(5,7,9)$

The conjugacy classes of H are

$D1 = Id(G)$

$D2 = (1,6,5)(2,7,3)(4,9,8)$

$D3 = (1,5,6)(2,3,7)(4,8,9)$

$D4 = (1,3,8)(2,4,6)(5,7,9)$

$D5 = (1,8,3)(2,6,4)(5,9,7)$

$D6 = (1,2,9)(3,4,5)(6,7,8)$

$D7 = (1,9,2)(3,5,4)(6,8,7)$

$D8 = (1,4,7)(2,5,8)(3,6,9)$

$D9 = (1,7,4)(2,8,5)(3,9,6)$

From the character tables of G and H we are going to use the information we labeled as ϕ of H and ϕ^G of G. By the definition of induction we induce the character $\phi = \chi.2$ of H up to $\phi^G = \chi.9$ of G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{72}{9} = 8$ and h_α is the size of $\chi.9$ in Table 5.3.

$\phi_1^G = \frac{8}{1}\sum_{w\in H\cap C_1}\phi(w)$

which implies $\phi_1^G = \frac{8}{1}(\phi(1)) = 8(1) = 8.$

$\phi_2^G = \frac{8}{9}\sum_{w\in H\cap C_2}\phi(w)$

$\implies \phi_2^G = \frac{8}{9}\sum_{w\in H\cap C_2}\phi(w)$=$\frac{8}{9}(0) = 0$ (since $H\cap C_2 = \phi$)

$\phi_3^G = \frac{8}{8}\sum_{w\in H\cap C_3}\phi(w)$

$\implies \phi_3^G = 1(1\phi((1,2,9)(3,4,5)(6,7,8)) + 1\phi((1,3,8)(2,4,6)(5,7,9))$
$+ 1\phi((1,9,2)(3,5,4)(6,8,7) + 1\phi((1,6,5)(2,7,3)(4,9,8)) + 1\phi((1,7,4)$
$(2,8,5)(3,9,6)) + 1\phi((1,4,7)(2,5,8)(3,6,9)) + 1\phi((1,8,3)(2,6,4)(5,9,7))$
$+ 1\phi((1,5,6)(2,3,7)(4,8,9))) = Z1 + (-Z1-1) + 1 + 1$
$+ Z1 + (-Z1-1) + Z1 + (-Z1-1) = -1$

$\phi_4^G = \frac{8}{9}\sum_{w\in H\cap C_4}\phi(w)$

$\implies \phi_4^G = \frac{8}{9}\sum_{w\in H\cap C_4}\phi(w)$=$\frac{8}{9}(0) = 0$ (since $H\cap C_4 = \phi$)

$\phi_5^G = \frac{8}{9}\sum_{w\in H\cap C_5}\phi(w)$

$\implies \phi_5^G = \frac{8}{9}\sum_{w\in H\cap C_5}\phi(w)$=$\frac{8}{9}(0) = 0$ (since $H\cap C_5 = \phi$)

$\phi_6^G = \frac{8}{9}\sum_{w\in H\cap C_6}\phi(w)$

$\implies \phi_6^G = \frac{8}{9}\sum_{w\in H\cap C_6}\phi(w)$=$\frac{8}{9}(0) = 0$ (since $H\cap C_6 = \phi$)

$\phi_7^G = \frac{8}{9}\sum_{w\in H\cap C_7}\phi(w)$

$\implies \phi_7^G = \frac{8}{9}\sum_{w\in H\cap C_7}\phi(w)$=$\frac{8}{9}(0) = 0$ (since $H\cap C_7 = \phi$)

$\phi_8^G = \frac{8}{9}\sum_{w\in H\cap C_8}\phi(w)$

$\implies \phi_8^G = \frac{8}{9}\sum_{w\in H\cap C_8}\phi(w) = \frac{8}{9}(0) = 0$ (since $H \cap C_8 = \phi$)

$\implies \phi_9^G = \frac{8}{9}\sum_{w\in H\cap C_9}\phi(w) = \frac{8}{9}(0) = 0$ (since $H \cap C_9 = \phi$)

So $\phi \uparrow_H^G = (8, 0, -1, 0, 0, 0, 0, 0, 0)$ and we have verified that $\chi_{.2}$ of H induces $\chi_{.9}$ of G.

```
>CG[9];
( 8, 0, -1, 0, 0, 0, 0, 0, 0 )
```

Table 8.1: Character Table of G

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	1	1	-1	-1	-1	-1
$\chi_{.3}$	1	1	1	-1	-1	I	$-I$	I	$-I$
$\chi_{.4}$	1	1	1	-1	-1	$-I$	I	$-I$	I
$\chi_{.5}$	1	-1	1	I	$-I$	$Z1$	$Z1\#3$	-Z1	$-Z1\#3$
$\chi_{.6}$	1	-1	1	$-I$	I	$Z1\#3$	$Z1$	$-Z1\#3$	$-Z1$
$\chi_{.7}$	1	-1	1	-I	I	$-Z1\#3$	$-Z1$	$Z1\#3$	$Z1$
$\chi_{.8}$	1	-1	1	I	$-I$	$-Z1$	$-Z1\#3$	$Z1$	$Z1\#3$
$\chi_{.9}$	8	0	-1	0	0	0	0	0	0

# denotes algebraic conjugation.
$Z1$ is the primitive three root of unity. I is Root of Unity four.

Table 8.2: Character Table of H

χ	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	J	$-1-J$	1	1	J	$-J-1$	J	$-1-J$
$\chi_{.3}$	1	$-1-J$	J	1	1	$-1-J$	$J-1$	-J	J
$\chi_{.4}$	1	1	1	J	$-1-J$	$J-1$	$-J-1$	$-J$	J
$\chi_{.5}$	1	J	$-1-J$	$J-1$	$-J-1$	$-J$	J	1	1
$\chi_{.6}$	1	$-1-J$	J	$J-1$	$-J$	1	1	$J-1$	-J
$\chi_{.7}$	1	1	1	$-1-J$	$J-1$	$-J$	J	$J-1$	-J
$\chi_{.8}$	1	$J-1$	-J-1	$-J$	J	1	1	$-1-J$	J
$\chi_{.9}$	1	$-1-J$	$J-1$	$-J$	J	J-1	$-J$	1	1

Before we verify the monomial representation of the matrices we must find the exact value of $Z1$. Noticed the order $| < Z > | = 3$ that means we must find the GFq(field) $\ni \frac{3}{p-1}$ has a subgroup isomorphic to $< Z >$ prime $p \ni \frac{3}{p-1}$. Implies $p = 7$. Since $\frac{3}{7-1} = \frac{3}{6}$. Thus $Z_7 - \{0\} =< 2 >$ since $2^6 = 1$ mod 7. We apply the following formula: $|a^k| = \frac{|a|}{gcd(k,|a|)}$ where $a = 2$. Since the generator of the cyclotomic field 7 is 2 and $|2| = 6$.
Thus,

$$|2^k| = \frac{|2|}{gcd(k, |2|)}$$

$$|2^k| = \frac{6}{gcd(k, 6)}$$

where $|2^k| = 3$.

Hence, $k = 2, 4$.

$$|4| = |2^2| = 3$$

$$|16| = |2^4| = 3$$

Take the first order $Z = 4$ where $k^a = 2^4 = 2$ mod 7 Now we can replace Z1 with 2 to calculate every entry of $CG[9], CH[2]$, and matrix.

Table 8.3: $\chi_{.9}$ of G

ϕ^G	Class	Size	Class Representative
8	C_1	1	Id(G)
0	C_2	9	(1, 9)(3, 7)(4, 6)(5, 8)
-1	C_3	8	(1, 2, 9)(3, 4, 5)(6, 7, 8)
0	C_4	9	(1, 7, 9, 3)(4, 5, 6, 8)
0	C_5	9	(1, 3, 9, 7)(4, 8, 6, 5)
0	C_6	9	(1, 5, 7, 6, 9, 8, 3, 4)
0	C_7	9	(1, 6, 3, 5, 9, 4, 7, 8)
0	C_8	9	(1, 8, 7, 4, 9, 5, 3, 6)
0	C_9	9	(1, 4, 3, 8, 9, 6, 7, 5)

Table 8.4: $\chi_{.2}$ of H

ϕ	Class	Size	Class Representative
1	D_1	1	Id(H)
$Z1$	D_2	1	(1, 6, 5)(2, 7, 3)(4, 9, 8)
$-Z1-1$	D_3	1	(1, 5, 6)(2, 3, 7)(4, 8, 9)
1	D_4	1	(1, 3, 8)(2, 4, 6)(5, 7, 9)
1	D_5	1	(1, 8, 3)(2, 6, 4)(5, 9, 7)
$Z1$	D_6	1	(1, 2, 9)(3, 4, 5)(6, 7, 8)
$-Z1-1$	D_7	1	(1, 9, 2)(3, 5, 4)(6, 8, 7)
$Z1$	D_8	1	(1, 4, 7)(2, 5, 8)(3, 6, 9)
$-Z1-1$	D_9	1	(1, 7, 4)(2, 8, 5)(3, 9, 6)

Equivalencies from table to construct matrix: $1 = 1$, $Z_1 = 2$.

Now prove the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} Z1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -Z1-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Z1-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Z1-1 \end{bmatrix},$$

and

$$A(zz) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Verifying the Monomial Representation

Since we have a linear character ϕ of the subgroup H of index 8 in G we let $G = Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4 \cup Ht_5 \cup Ht_6 \cup Ht_7 \cup Ht_8$ where the $t_i's$ are transversals of G acting on H.

That is G = He $\cup$ $H(1,6,4,5,2,3,8,7) \cup H(1,4,2,8)(3,7,6,5) \cup$
$H(1,5,8,6,2,7,4,3) \cup H(1,2)(3,6)(4,8)(5,7) \cup H(1,3,4,7,2,6,8,5)$
$\cup\ H(1,8,2,4)(3,5,6,7) \cup H(1,7,8,3,2,5,4,6)$.

Continuing the process in a 8x8 matrix:

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \phi(t_1xt_3^{-1}) & ... & \phi(t_1xt_7^{-1}) & \phi(t_1xt_8^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \phi(t_2xt_3^{-1}) & ... & \phi(t_2xt_7^{-1}) & \phi(t_2xt_8^{-1}) \\ \phi(t_3xt_1^{-1}) & \phi(t_3xt_2^{-1}) & \phi(t_3xt_3^{-1}) & ... & \phi(t_3xt_7^{-1}) & \phi(t_3xt_8^{-1}) \\ ... & \phi(t_4xt_2^{-1}) & \phi(t_4xt_3^{-1}) & ... & \phi(t_4xt_7^{-1}) & \phi(t_4xt_8^{-1}) \\ ... & \phi(t_5xt_2^{-1}) & \phi(t_5xt_3^{-1}) & ... & \phi(t_5xt_7^{-1}) & \phi(t_5xt_8^{-1}) \\ ... & \phi(t_6xt_2^{-1}) & \phi(t_6xt_3^{-1}) & ... & \phi(t_6xt_7^{-1}) & \phi(t_6xt_8^{-1}) \\ \phi(t_7xt_1^{-1}) & \phi(t_7xt_2^{-1}) & \phi(t_7xt_3^{-1}) & ... & \phi(t_7xt_7^{-1}) & \phi(t_7xt_8^{-1}) \\ \phi(t_8xt_1^{-1}) & \phi(t_8xt_2^{-1}) & \phi(t_8xt_3^{-1}) & ... & \phi(t_8xt_7^{-1}) & \phi(t_8xt_8^{-1}) \end{bmatrix}$$

We will calculate all 8 elements of the matrix using the following calculation where $t_1 = e,\ t_2 = (1,6,4,5,2,3,8,7),\ t_3 = (1,4,2,8)(3,7,6,5),\ t_4 = (1,5,8,6,2,7,4,3),$ $t_5 = (1,2)(3,6)(4,8)(5,7),\ t_6 = (1,3,4,7,2,6,8,5),\ t_7 = (1,8,2,4)(3,5,6,7),\ t_8 =$ $(1,7,8,3,2,5,4,6)$ and $x = (1,2,9)(3,4,5)(6,7,8)$ with each t_i corresponding to each transversal respectively from the G listed above. We will start by calculating the multiplication of t_ixt_j^{-1}.
$\phi(t_1xt_1^{-1}) = \phi(e(1,2,9)(3,4,5)(6,7,8)e^-1) = \phi((1,2,9)(3,4,5)(6,7,8))$. We will go directly to what $\phi(t_1xt_1^{-1})$ is equal to. Since $(1,2,9)(3,4,5)(6,7,8) \in H$. We look back to our Table 5.4 noticed
$\phi((1,2,9)(3,4,5)(6,7,8))$ is in class D_6=$Z1$. Therefore, the nonzero entry for *Row* 1 is $Z1$. We proceed as follows:

Row 1:

$\phi(t_1xt_1^{-1}) = \phi((1,2,9)(3,4,5)(6,7,8)) = Z1 = 2$

(since $\phi((1,2,9)(3,4,5)(6,7,8)) \in D_6$)

$\phi(t_1xt_2^{-1}) = \phi((1,5,2,9,7,3,6,8)) = 0$

$\phi(t_1xt_3^{-1}) = \phi((1,4,6,3)(2,9,8,7)) = 0$

$\phi(t_1xt_4^{-1}) = \phi((1,6,2,9,3,7,5,4)) = 0$

$\phi(t_1xt_5^{-1}) = \phi((2,9)(3,8)(4,7)(5,6)) = 0$

$\phi(t_1xt_6^{-1}) = \phi((1,7,6,4,8,2,9,5)) = 0$

$\phi(t_1xt_7^{-1}) = \phi((1,8,5,7)(2,9,4,3)) = 0$

$\phi(t_1xt_8^{-1}) = \phi((1,3,5,8,4,2,9,6)) = 0$

Row 2:

$\phi(t_2xt_1^{-1}) = \phi((1,7,2,4,3,6,5,9)) = 0$

$\phi(t_2xt_2^{-1}) = \phi((1,8,3)(2,6,4)(5,9,7)) = 1$

(since $\phi((1,8,3)(2,6,4)(5,9,7)) \in D_5$)

$\phi(t_2xt_3^{-1}) = \phi((1,3,7,4,5,9,8,2)) = 0$

$\phi(t_2xt_4^{-1}) = \phi((1,2,7,6)(3,8,5,9)) = 0$

$\phi(t_2xt_5^{-1}) = \phi((1,5,9,2,8,4,6,7)) = 0$

$\phi(t_2xt_6^{-1}) = \phi((1,4)(2,3)(5,9)(6,8)) = 0$

$\phi(t_2xt_7^{-1}) = \phi((1,6,3,5,9,4,7,8)) = 0$

$\phi(t_2xt_8^{-1}) = \phi((2,5,9,6)(3,4,8,7)) = 0$

Row 3:

$\phi(t_3xt_1^{-1}) = \phi((1,5,4,9)(2,6,3,8)) = 0$

$\phi(t_3xt_2^{-1}) = \phi((1,4,9,7,8,5,6,2)) = 0$

$\phi(t_3xt_3^{-1}) = \phi((1,6,5)(2,7,3)(4,9,8)) = Z1 = 2$

(since $\phi((1,6,5)(2,7,3)(4,9,8) \in D_2$)

$\phi(t_3xt_4^{-1}) = \phi((2,8,6,4,9,3,5,7)) = 0$

$\phi(t_3xt_5^{-1}) = \phi((1,7,5,8)(2,3,4,9)) = 0$

$\phi(t_3xt_6^{-1}) = \phi((1,8,7,4,9,5,3,6)) = 0$

$\phi(t_3xt_7^{-1}) = \phi((1,3)(2,5)(4,9)(6,7)) = 0$

$\phi(t_3xt_8^{-1}) = \phi((1,2,4,9,6,8,3,7)) = 0$

Row 4:

$\phi(t_4xt_1^{-1}) = \phi((1,3,2,8,7,5,6,9)) = 0$

$\phi(t_4xt_2^{-1}) = \phi((1,2,3,5)(4,6,9,7)) = 0$

$\phi(t_4xt_3^{-1}) = \phi((1,5,7,6,9,8,3,4)) = 0$

$\phi(t_4xt_4^{-1}) = \phi((1,4,7)(2,5,8)(3,6,9)) = Z1 = 2$

(since $\phi((1,4,7)(2,5,8)(3,6,9) \in D_8)$

$\phi(t_4xt_5^{-1}) = \phi((1,6,9,2,4,8,5,3)) = 0$

$\phi(t_4xt_6^{-1}) = \phi((2,6,9,5)(3,7,8,4)) = 0$

$\phi(t_4xt_7^{-1}) = \phi((1,7,3,8,6,9,4,2)) = 0$

$\phi(t_4xt_8^{-1}) = \phi((1,8)(2,7)(4,5)(6,9)) = 0$

Row 5:

$\phi(t_5xt_1^{-1}) = \phi((1,9)(3,7)(4,6)(5,8)) = 0$

$\phi(t_5xt_2^{-1}) = \phi((1,9,7,2,5,3,8,4)) = 0$

$\phi(t_5xt_3^{-1}) = \phi((1,9,8,6)(2,4,7,5)) = 0$

$\phi(t_5xt_4^{-1}) = \phi((1,9,3,2,6,7,4,8)) = 0$

$\phi(t_5xt_5^{-1}) = \phi((1,9,2)(3,5,4)(6,8,7)) = -Z1 - 1 = -2 - 1 = 4 \ mod \ 7$

(since $\phi((1,9,2)(3,5,4)(6,8,7) \in D_7)$

$\phi(t_5xt_6^{-1}) = \phi((1,9,5,6,3,4,2,7)) = 0$

$\phi(t_5xt_7^{-1}) = \phi((1,9,4,5)(2,8,3,6)) = 0$

$\phi(t_5xt_8^{-1}) = \phi((1,9,6,5,7,8,2,3)) = 0$

Row 6:

$\phi(t_6xt_1^{-1}) = \phi((1,4,8,3,5,2,7,9)) = 0$

$\phi(t_6xt_2^{-1}) = \phi((1,6)(2,8)(3,4)(7,9)) = 0$

$\phi(t_6xt_3^{-1}) = \phi((2,3,6,7,9,8,5,4)) = 0$

$\phi(t_6xt_4^{-1}) = \phi((1,7,9,3)(4,5,6,8)) = 0$

$\phi(t_6xt_5^{-1}) = \phi((1,8,6,3,7,9,2,5)) = 0$

$\phi(t_6xt_6^{-1}) = \phi((1,3,8)(2,4,6)(5,7,9)) = 1$
(since $\phi((1,3,8)(2,4,6)(5,7,9) \in D_4)$
$\phi(t_6xt_7^{-1}) = \phi((1,2,6,5,8,7,9,4)) = 0$
$\phi(t_6xt_8^{-1}) = \phi((1,5,3,2)(4,7,9,6)) = 0$

Row 7:
$\phi(t_7xt_1^{-1}) = \phi((1,6,8,9)(2,5,7,4)) = 0$
$\phi(t_7xt_2^{-1}) = \phi((2,4,5,8,9,7,6,3)) = 0$
$\phi(t_7xt_3^{-1}) = \phi((1,7)(2,6)(3,5)(8,9)) = 0$
$\phi(t_7xt_4^{-1}) = \phi((1,8,9,3,4,6,5,2)) = 0$
$\phi(t_7xt_5^{-1}) = \phi((1,3,6,4)(2,7,8,9)) = 0$
$\phi(t_7xt_6^{-1}) = \phi((1,2,8,9,5,4,7,3)) = 1$
$\phi(t_7xt_7^{-1}) = \phi((1,5,6)(2,3,7)(4,8,9)) = -Z1 - 1 = 4 \; mod \; 7$
(since $\phi((1,5,6)(2,3,7)(4,8,9) \in D_3)$
$\phi(t_7xt_8^{-1}) = \phi((1,4,3,8,9,6,7,5)) = 0$

Row 8:
$\phi(t_8xt_1^{-1}) = \phi((1,8,4,7,6,2,3,9)) = 0$
$\phi(t_8xt_2^{-1}) = \phi((1,3,9,7)(4,8,6,5)) = 0$
$\phi(t_8xt_3^{-1}) = \phi((1,2,5,6,4,3,9,8)) = 0$
$\phi(t_8xt_4^{-1}) = \phi((1,5)(2,4)(3,9)(7,8)) = 0$
$\phi(t_8xt_5^{-1}) = \phi((1,4,5,7,3,9,2,6)) = 0$
$\phi(t_8xt_6^{-1}) = \phi((1,6,7,2)(3,9,5,8)) = 1$
$\phi(t_8xt_7^{-1}) = \phi((2,7,5,3,9,4,6,8)) = 0$
$\phi(t_8xt_8^{-1}) = \phi((1,7,4)(2,8,5)(3,9,6)) = -Z1 - 1 = 4 \; mod \; 7$
(since $\phi((1,7,4)(2,8,5)(3,9,6) \in D_9)$

Each ϕ of H corresponded with a conjugacy class of either H or G. If the element is in a conjugacy class from H (seen in table 5.4) we write the value of ϕ for that class. Since our matrix was produced in cyclotomic field 7, we needed to produce an order 3 element in Z_7. In this case, 2 was chosen as the element of order 3. To complete this process, the matrix for zz should also be verified by repeating the process above where

$zz = (1, 6, 4, 5, 2, 3, 8, 7)$ and matrix

$$A(zz) = \begin{bmatrix} \phi(t_1 z t_1^{-1}) & \phi(t_1 z t_2^{-1}) & \phi(t_1 z t_3^{-1}) & ... & \phi(t_1 z t_7^{-1}) & \phi(t_1 z t_8^{-1}) \\ \phi(t_2 z t_1^{-1}) & \phi(t_2 z t_2^{-1}) & \phi(t_2 z t_3^{-1}) & ... & \phi(t_2 z t_7^{-1}) & \phi(t_2 z t_8^{-1}) \\ \phi(t_3 z t_1^{-1}) & \phi(t_3 z t_2^{-1}) & \phi(t_3 z t_3^{-1}) & ... & \phi(t_3 z t_7^{-1}) & \phi(t_3 z t_8^{-1}) \\ ... & \phi(t_4 z t_2^{-1}) & \phi(t_4 z t_3^{-1}) & ... & \phi(t_4 z t_7^{-1}) & \phi(t_4 z t_8^{-1}) \\ ... & \phi(t_5 z t_2^{-1}) & \phi(t_5 z t_3^{-1}) & ... & \phi(t_5 z t_7^{-1}) & \phi(t_5 z t_8^{-1}) \\ ... & \phi(t_6 z t_2^{-1}) & \phi(t_6 z t_3^{-1}) & ... & \phi(t_6 z t_7^{-1}) & \phi(t_6 z t_8^{-1}) \\ \phi(t_7 z t_1^{-1}) & \phi(t_7 z t_2^{-1}) & \phi(t_7 z t_3^{-1}) & ... & \phi(t_7 z t_7^{-1}) & \phi(t_7 z t_8^{-1}) \\ \phi(t_8 z t_1^{-1}) & \phi(t_8 z t_2^{-1}) & \phi(t_8 z t_3^{-1}) & ... & \phi(t_8 z t_7^{-1}) & \phi(t_8 z t_8^{-1}) \end{bmatrix}.$$

Repeating the same process from above we get the following calculations for each row of the matrix $A(zz)$.

Row 1:

$\phi(t_1 z t_1^{-1}) = \phi((1, 6, 4, 5, 2, 3, 8, 7)) = 0$
$\phi(t_1 z t_2^{-1}) = \phi(e) = 1$
$\phi(t_1 z t_3^{-1}) = \phi((1, 7, 8, 3, 2, 5, 4, 6)) = 0$
$\phi(t_1 z t_4^{-1}) = \phi((1, 8, 2, 4)(3, 5, 6, 7)) = 0$
$\phi(t_1 z t_5^{-1}) = \phi((1, 3, 4, 7, 2, 6, 8, 5)) = 0$
$\phi(t_1 z t_6^{-1}) = \phi((1, 2)(3, 6)(4, 8)(5, 7)) = 0$
$\phi(t_1 z t_7^{-1}) = \phi((1, 5, 8, 6, 2, 7, 4, 3)) = 0$
$\phi(t_1 z t_8^{-1}) = \phi((1, 4, 2, 8)(3, 7, 6, 5)) = 0$

Row 2:

$\phi(t_2 z t_1^{-1}) = \phi((1, 4, 2, 8)(3, 7, 6, 5)) = 0$
$\phi(t_2 z t_2^{-1}) = \phi((1, 6, 4, 5, 2, 3, 8, 7)) = 0$
$\phi(t_2 z t_3^{-1}) = \phi(e) = 1$
$\phi(t_2 z t_4^{-1}) = \phi((1, 7, 8, 3, 2, 5, 4, 6)) = 0$
$\phi(t_2 z t_5^{-1}) = \phi((1, 8, 2, 4)(3, 5, 6, 7)) = 0$
$\phi(t_2 z t_6^{-1}) = \phi((1, 3, 4, 7, 2, 6, 8, 5)) = 0$
$\phi(t_2 z t_7^{-1}) = \phi((1, 2)(3, 6)(4, 8)(5, 7)) = 0$

$\phi(t_2zt_8^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$

Row 3:
$\phi(t_3zt_1^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$
$\phi(t_3zt_2^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$
$\phi(t_3zt_3^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$
$\phi(t_3zt_4^{-1}) = \phi(e) = 1$
$\phi(t_3zt_5^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$
$\phi(t_3zt_6^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$
$\phi(t_3zt_7^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$
$\phi(t_3zt_8^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$

Row 4:
$\phi(t_4zt_1^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$
$\phi(t_4zt_2^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$
$\phi(t_4zt_3^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$
$\phi(t_4zt_4^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$
$\phi(t_4zt_5^{-1}) = \phi(e) = 1$
$\phi(t_4zt_6^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$
$\phi(t_4zt_7^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$
$\phi(t_4zt_8^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$

Row 5:
$\phi(t_5zt_1^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$
$\phi(t_5zt_2^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$
$\phi(t_5zt_3^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$
$\phi(t_5zt_4^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$
$\phi(t_5zt_5^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$
$\phi(t_5zt_6^{-1}) = \phi((e) = 1$
$\phi(t_5zt_7^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$
$\phi(t_5zt_8^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$

Row 6:

$\phi(t_6 z t_1^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$

$\phi(t_6 z t_2^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$

$\phi(t_6 z t_3^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$

$\phi(t_6 z t_4^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$

$\phi(t_6 z t_5^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$

$\phi(t_6 z t_6^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$

$\phi(t_6 z t_7^{-1}) = \phi(e) = 1$

$\phi(t_6 z t_8^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$

Row 7:

$\phi(t_7 z t_1^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$

$\phi(t_7 z t_2^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$

$\phi(t_7 z t_3^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$

$\phi(t_7 z t_4^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$

$\phi(t_7 z t_5^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$

$\phi(t_7 z t_6^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$

$\phi(t_7 z t_7^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$

$\phi(t_7 z t_8^{-1}) = \phi(e) = 1$

Row 8:

$\phi(t_8 z t_1^{-1}) = \phi(e) = 1$

$\phi(t_8 z t_2^{-1}) = \phi((1,7,8,3,2,5,4,6)) = 0$

$\phi(t_8 z t_3^{-1}) = \phi((1,8,2,4)(3,5,6,7)) = 0$

$\phi(t_8 z t_4^{-1}) = \phi((1,3,4,7,2,6,8,5)) = 0$

$\phi(t_8 z t_5^{-1}) = \phi((1,2)(3,6)(4,8)(5,7)) = 0$

$\phi(t_8 z t_6^{-1}) = \phi((1,5,8,6,2,7,4,3)) = 0$

$\phi(t_8 z t_7^{-1}) = \phi((1,4,2,8)(3,7,6,5)) = 0$

$\phi(t_8 z t_8^{-1}) = \phi((1,6,4,5,2,3,8,7)) = 0$

Therefore, we have proven the monomial representation has the generators

$$A(xx) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

and

$$A(zz) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $|A(xx)| = 3$ and $|A(zz)| = 8$ with $|A(xx) \cdot A(zz)| = 8$, the order of our index, we know that $\langle A(xx)$ and $A(zz)\rangle$ is a faithful representation of G. We proceed and produce a permutation representation.

Constructing a Permutation Representation

We worked in Z_7 on matrices of degree 8x8 thus we are produce a $7^{*8} :_m (3^2 : 8)$ progenitor permutation representation based on the monomial representation $(3^2 : 8) = \langle A(xx), A(zz)\rangle$. Consider the matrix entries for $a_{i,j}$ for A(xx) and A(zz). We know there are 8 t's, one for each column of our matrix and because we have a semi-direct product in our progenitor then the elements of $(3^2 : 8)$ act as automorphisms of $< t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 > * < t_6 > * < t_7 > * < t_8 >$.

Now we find the permutations for our two matrices xx and zz, to do so we use the following formula

$$a_{i,j} = 1 \text{ if the automorphism takes } t_i \to t_j$$
$$a_{i,j} = n \text{ if the automorphism takes } t_i \to t_j^n.$$

Since 7^{*8} is a free product of two cyclic groups of order 6 we will construct a table with 8 t's of order 6 labeled from 1...48 found in table 5.5. Viewing the entries of A(xx) $a_{11} = 2$, $a_{22} = 1$, $a_{33} = 2$, $a_{44} = 2$, $a_{55} = 4$, $a_{66} = 1$, $a_{77} = 4$, and $a_{88} = 4$.

Thus,

$t_1 \to t_1^2$,

$t_2 \to t_2^1$,

$t_3 \to t_3^2$,

$t_4 \to t_4^2$,

$t_5 \to t_5^4$,

$t_6 \to t_6^1$,

$t_7 \to t_7^4$,

$t_8 \to t_8^4$

Proceeding on finding a permutation representative we must describe Table 5.5. The top number of the chart labels each element, the middle of the chart shows what the automorphism produces, and the bottom number shows what that produced element is numbered on the top of the chart. The same process is conducted for generators zz, but for Table 5.6 we will only show part of the process.

Table 8.5: Automorphisms of A(xx)

1	2	3	4	5	6	7	8
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^2	t_2	t_3^2	t_4^2	t_5^4	t_6	t_7^4	t_8^4
9	26	11	12	29	6	31	32

9	10	11	12	13	14	15	16
t_1^2	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_2^7	t_2^8
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^4	t_2^2	t_3^4	t_4^4	t_5	t_6^2	t_7^1	t_8
25	10	27	28	5	14	7	8

17	18	19	20	21	22	23	24
t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_7^3	t_8^3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^6	t_2^3	t_3^6	t_4^6	t_5^5	t_6^3	t_7^5	t_8^5
41	18	43	44	37	22	39	40

25	26	27	28	29	30	31	32
t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4	t_7^4	t_8^4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_2^4	t_3^1	t_4^1	t_5^2	t_6^4	t_7^2	t_8^2
1	26	3	4	13	30	15	16

33	34	35	36	37	38	39	40
t_1^5	t_2^5	t_3^5	t_4^5	t_5^5	t_6^5	t_7^5	t_8^5
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^3	t_2^5	t_3^3	t_4^3	t_5^6	t_6^5	t_7^6	t_8^6
17	34	19	20	45	38	47	48

41	42	43	44	45	46	47	48
t_1^6	t_2^6	t_3^6	t_4^6	t_5^6	t_6^6	t_7^6	t_8^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^5	t_2^6	t_3^5	t_4^5	t_5^3	t_6^6	t_7^3	t_8^3
33	42	35	36	21	46	23	24

Viewing the nonzero entries from the monomial representation, $A(zz)$, and using the formula listed above we have $a_{12} = 1$, $a_{23} = 1$, $a_{34} = 1$, $a_{45} = 1$, $a_{56} = 1$, $a_{67} = 1$, $a_{78} = 1$, and $a_{81} = 1$.

Thus,

$t_1 \to t_2^1$,

$t_2 \to t_3^1$,

$t_3 \to t_4^1$,

$t_4 \to t_5^1$,

$t_5 \to t_6^1$,

$t_6 \to t_7^1$,

$t_7 \to t_8^1$,

$t_8 \to t_1^1$.

Table 8.6: Automorphisms of A(zz)

1	2	3	4	5	6	7	8	...	47	48
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	...	t_7^6	t_8^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_1	...	t_8^6	t_1^6
1	2	3	4	5	6	7	8	...	47	48

From Table 5.5 and 5.6, we can construct permutations for $A(xx)$ and $A(zz)$ by using our labels from each automorphism. Consider element #1 be denoted as t_1 from Table 5.5. This produces the permutation $(1, 9, 25)$. If we follow each element and its corresponding automorphism number labeling and repeating the process for xx, we produce the following permutations:

$A(xx) = (1, 9, 25)$, $(17, 41, 33)$, $(3, 11, 27)$, $(19, 43, 35)$, $(4, 12, 28)$, $(20, 44, 36)$, $(5, 29, 13)$, $(21, 37, 45)$ $(7, 31, 15)$, $(23, 39, 47)$, $(8, 32, 16)$, $(24, 40, 48)$

Likewise for zz.

$A(zz) = (1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12, 13, 14, 15, 16),$
$(17, 18, 19, 20, 21, 22, 23, 24)(25, 26, 27, 28, 29, 30, 31, 32),$
$(33, 34, 35, 36, 37, 38, 39, 40)(41, 42, 43, 44, 45, 46, 47, 48)$
Therefore, we have constructed a permutation representation from our matrices.

Creating a Presentation of the Progenitor

To construct a presentation for the progenitor we must choose a t to normalize from our eight choices $< t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 > * < t_6 > * < t_7 > * < t_8 >$. Let $t \sim t_1$. Now we find permutations which normalizes $< t_1 >$ or fixes the following set
$\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6\}$. This is a defining characteristic of a monomial progenitors. Monomial progenitors fix a set of $t's$ while permutation progenitors fix only one specific t_i. Using MAGMA we were able to find

```
> Sch:=SchreierSystem(G,sub<G|Id(G)>);
>  ArrayP:=[Id(N): i in [1..#N]];
>  for i in [2..#N] do
for>  P:=[Id(N): l in [1..#Sch[i]]];
for>  for j in [1..#Sch[i]] do
for|for>  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for>  if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
for|for>  if Eltseq(Sch[i])[j] eq 2 then P[j]:=zz; end if;
for|for>  if Eltseq(Sch[i])[j] eq -2 then P[j]:=zz^-1; end if;
for|for>  end for;
for>  PP:=Id(N);
for>  for k in [1..#P] do
for|for>  PP:=PP*P[k]; end for;
for>  ArrayP[i]:=PP;
for>  end for;
> Normaliser:=Stabiliser(N,{1,9,17,25,33,41});

> Stabiliser(N,{1,9,17,25,33,41});
 (2, 10, 26)(3, 11, 27)(4, 28, 12)(6, 30, 14)(7, 31, 15)
 (8, 16, 32)(18, 42,34)(19, 43, 35)(20, 36, 44)
 (22, 38, 46)(23, 39, 47)(24, 48, 40)(1, 25, 9)(3, 27, 11)
 (4, 28, 12)(5, 13, 29)(7, 15, 31)
 (8, 16, 32)(17, 33,41)(19, 35, 43)(20, 36, 44)(21, 45, 37)
 (23, 47, 39)(24, 48, 40)
```

```
>A1:=N!(2, 10, 26)(3, 11, 27)(4, 28, 12)(6, 30, 14)(7, 31, 15)
>(8, 16, 32)(18, 42,34)(19, 43, 35)(20, 36, 44)(22, 38, 46)
>(23, 39, 47)(24, 48, 40);

B1:=N! (1, 25, 9)(3, 27, 11)(4, 28, 12)(5, 13, 29)(7, 15, 31)
>(8, 16, 32)(17, 33,41)(19, 35, 43)(20, 36, 44)(21, 45, 37)
>(23, 47, 39)(24, 48, 40);

> for i in [1..72] do if ArrayP[i] eq A1
then Sch[i]; end if; end for;
x

>for i in [1..72] do if ArrayP[i] eq B1
 then Sch[i]; end if; end for;
x * (z^4*x*z^2*x*z^2)^-1
```

For a presentation we must convert these permutations into words which we find to be: A1 $= x$, x stabilises the $t's$ while B1 $=x*(z^4*x*z^2*x*z^2)^{-1}$ commutes with the $t's$.

We look at the permutation $xx = (1,9,25)$,$(17,41,33)$, $(3,11,27)$, $(19,43,35),...$ so x stabilises 1 which is 9. We check the label in our Table 5.5. If the element is 2, then the automorphism is t_1^2

$$\text{A1} \to t^x = t_1^2.$$

So, a presentation for the progenitor of $7^{*8} :_m (3^2 : 8)$ is $G < x,z,t > := Group <$ $x,z,t|x^3, (x, z^4xz^2xz^2), (z^4xz^2xz^2)^{-1}z^{-1}x^{-1}z,$
$z(z^4xz^2xz^2)^{-1}x^{-1}z^{-1}(z^4xz^2xz^2)^{-1}, z^8, t^7, (t, x(z^4xz^2xz^2)^{-1}), t^x = t^2 >;$

Now we check if our progenitor is correct. We apply Grindstaff's Lemma. Our symmetric generators are t_1, t_2, t_3, t_4, t_5, and t_6. We want to add to the above presentation that all $ti's$ commute; that is, (t_1,t_2),(t_1,t_3),(t_1,t_4), (t_1,t_5), (t_1,t_6), (t_1,t_7), and (t_1,t_8). Now $t_2 = t^z$, $t_3 = t^{z^2}$, $t_4 = t^{z^3}$, $t_5 = t^{z^4}$, and $t_6 = t^{z^5}$.

So we check if

$7^{*8} :_m (3^2 : 8) = G < x,z,t > := Group < x,z,t|x^3, (z^4xz^2xz^2)^3, (x, z^4xz^2xz^2),$
$(z^4xz^2xz^2)^{-1}z^{-1}x^{-1}z, z(z^4xz^2xz^2)^{-1}x^{-1}z^{-1}(z^4xz^2xz^2)^{-1}, z^8, t^7, (t, x(z^4xz^2xz^2)^{-1}),$
$t^x = t^2 >$; factored by $t_2 = t^z$, $t_3 = t^{z^2}$, $t_4 = t^{z^3}$, $t_5 = t^{z^4}$, $t_6 = t^{z^5}$, $t_7 = t^{z^6}$, and $t_8 = t^{z^7}$

is the group $7^8 :_m (3^2 : 8)$ of order $7^8 \times (3^2 \times 8)$.

```
 G<x,z,t>:=Group<x,z,t|x^3,(z^4*x*z^2*x*z^2)^3,
 (x,z^4*x*z^2*x*z^2),
(z^4*x*z^2*x*z^2)^-1*z^-1*x^-1*z,
z*(z^4*x*z^2*x*z^2)^-1*x^-1*z^-1*(z^4*x*z^2*x*z^2)^-1,z^8,
t^7,
(t,x * (z^4*x*z^2*x*z^2)^-1),t^x=t^2,
(t,t^z),(t,t^(z^2)),(t,t^(z^3)),(t,t^(z^4)),(t,t^(z^5)),
(t,t^(z^6)),(t,t^(z^7))>;
   print Index(G,sub<G|x,z>: CosetLimit:=9^10,
   Hard:=true, Print:=2);
```

Grinstaff Lemma is verified, the index matches the order of $7^8 \times (3^2 \times 8)$=415065672. Next, we find finite homomorphic images of the progenitor $7^{*8} :_m (3^2 : 8)$. Thus, we factor the progenitor by additional relations. Here, we have factored by first order relations.

```
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^3,(x,y),y^-1*z^-1*x^-1*z,
z*y^-1*x^-1*z^-1*y^-1,z^8,
t^7,
(t,x * y^-1),t^(x)=t^2, (z^4*t)^a,(y*t)^b, (z^2*t)^c,
(z^-2*t)^d,(z*t)^e, (z^3*t)^f,(z^-3*t)^g, (z^-1*t)^h>;
if #G gt 72 then
#G, a,b,c,d,e,f,g,h;
end if;
end for;
```

Progenitors with no images due to MAGMA resources of $7^{*8} :_m (3^2 : 8)$ SInce we do not get finite homomorphic images for this group, now we are going to show the isomorphism type of the control group N. We used the following codes to prove the isomorphism type $3^2 : 8$. The CompositionFactor and the NormalLattice of the group G is listed as:

```
> G:=TransitiveGroup(9,15);
>CompositionFactors(G);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
```

```
     *
     |  Cyclic(3)
     *
     |  Cyclic(3)
     1
> NL:=NormalLattice(G);
> NL;

Normal subgroup lattice
-----------------------

[5]  Order 72  Length 1  Maximal Subgroups: 4
---
[4]  Order 36  Length 1  Maximal Subgroups: 3
---
[3]  Order 18  Length 1  Maximal Subgroups: 2
---
[2]  Order 9   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
```

We continue by finding the largest abelian group. In this case the largest abelian group is the subgroup $NL[2]$ of order 9. We verify by asking MAGMA the following:

```
> IsAbelian(NL[2]);
true
```

Since $NL[2]$ is the largest abelian group, then we verify if it is 3×3, 3^2, or 9.

```
> X:=AbelianGroup(GrpPerm,[3,3]);
> IsIsomorphic(NL[2],X);
true Mapping from: GrpPerm: $, Degree 9,
Order 3^2 to GrpPerm: X
```

By $MAGMA$, K is given in the form 3×3. Now we check what is Q. To do so, we factor Q by the largest abelian group, $NL[2]$ and labeled it q.

```
> q,ff:=quo<G|NL[2]>;
> q;
Permutation group q acting on a set of cardinality 8
Order = 8 = 2^3
     Id(q)
     Id(q)
     (1, 2, 3, 4, 5, 6, 7, 8)
```

Note q is of order 8.

Now we find the transversal of G and $NL[2]$ and store the transversal as $T[2]$.

```
> T:=Transversal(G,NL[2]);
> ff(T[2]) eq q!(1, 2, 3, 4, 5, 6, 7, 8);
true
```

Now find the permutations of $NL[2]$ and store them as A and B.

```
> NL[2].1;
(1, 6, 5)(2, 7, 3)(4, 9, 8)
> A:=G!(1, 6, 5)(2, 7, 3)(4, 9, 8);
> NL[2].2;
(1, 3, 8)(2, 4, 6)(5, 7, 9)
> B:=G!(1, 3, 8)(2, 4, 6)(5, 7, 9);
```

Now we write a presentation with a^3 and b^3 and since we have the semi-direct product, we check the action of $a^c =?$ and $b^c =?$. We have the following presentation with the unknown action of $a^c =?$ and $b^c =?$ of the monomial progenitor.

$$< a, b, c|a^3, b^3, (a, b), c^8, a^c =?, b^c =? >$$

Now we use Magma to find the action of $a^c =?$ and $b^c =?$.

```
> for i,j in [0..2] do if A^T[2] eq A^i*B^j
 then i,j; end if; end for;
0 2
> for i,j in [0..2] do if B^T[2] eq A^i*B^j
then i,j; end if; end for;
2 2
```

After we check the action of c on a and b, we get the following presentation:

$$H < a, b, c >:= Group < a, b, c|a^3, b^3, (a, b), c^8, a^c = b^2, b^c = a^2b^2 >;$$

To verify if the presentation of the isomorphism type of N is correct, we ask $MAGMA$ the following:

```
    > f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(G,H1);
true
```

Therefore, the isomorphism type of $N \cong 3^2 : 8$.

8.2 Monomial Progenitor $53^{*4} :_m (13 : 4)$

To construct a monomial presentation of $53^{*4} :_m 13 : 4$, first we must induce a linear character from a subgroup H of G. To insure we get an irreducible character we must choose a subgroup with index matching the degree of an irreducible character of G. We consider the character table of G in Table 5.1 and note G has characters $\chi_{.1}, \chi_{.2}, ..., \chi_{.7}$. We proceed using $\chi_{.5}$ and look for a subgroup of order 4 so that $\frac{|G|}{|H|} = \frac{52}{13} = 4$.
Since the index of the two groups is 4. This implies, if a matrix representation is faithful then $A(xx)$ and $A(yy)$ will be represented by a 4×4 matrices.

Verifying the Induction

We produce a character table for $\chi_{.5}$ in table 5.2. We will verify the induction $\chi_{.3}$ of H to $\chi_{.5}$ of G by considering the irreducible characters ϕ (of H) and ϕ^G (of G). G is generated by x and y, where $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$ and $y = (1, 5, 12, 8)(2, 10, 11, 3)(4, 7, 9, 6)$.

The Conjugacy classes of group G are
$C1 = Id(G)$
$C2 = (1, 3)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9), (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7), ...$ $C3 = (1, 3, 13, 11)(2, 8, 12, 6)(4, 5, 10, 9), (1, 4, 6, 3)(2, 9, 5, 11)(7, 8, 13, 12), ...$
$C4 = (1, 7, 3, 10)(4, 5, 13, 12)(6, 8, 11, 9), (1, 10, 4, 8)(2, 5, 3, 13)(6, 11, 12, 7), ...$
$C5 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13), (1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3, 11, 6), ...$
$C6 = (1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4), (1, 4, 7, 10, 13, 3, 6, 9, 12, 2, 5, 8, 11), ...$
$C7 = (1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5), (1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7), ...$

Consider the subgroup H of G given below.
$H = Id(G), (1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)$

The conjugacy classes of H are
$D1 = Id(G)$
$D2 = (1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)$

$D3 = (1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3, 11, 6)$
$D4 = (1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$
$D5 = (1, 4, 7, 10, 13, 3, 6, 9, 12, 2, 5, 8, 11)$
$D6 = (1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7)$
$D7 = (1, 12, 10, 8, 6, 4, 2, 13, 11, 9, 7, 5, 3)$
$D8 = (1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)$
$D9 = (1, 7, 13, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8)$
$D10 = (1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$
$D11 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$
$D12 = (1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9)$
$D13 = (1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5)$

From the character tables of G and H we are going to use only the information we labeled as ϕ (of H) and ϕ^G (of G). Using our definition of induction we induce the character $\phi = \chi.3$ of H up to $\phi^G = \chi.5$ of G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$
$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{52}{13} = 4$.

$\phi_1^G = \frac{4}{1} \sum_{w \in H \cap C_1} \phi(w)$

which implies $\phi_1^G = \frac{4}{1}(\phi(1)) = 4(1) = 4$.

$\phi_2^G = \frac{4}{13} \sum_{w \in H \cap C_2} \phi(w)$

$\implies \phi_2^G = \frac{4}{13} \sum_{w \in H \cap C_2} \phi(w)$=0 (since $H \cap C_2 = \phi$)

$\phi_3^G = \frac{4}{13} \sum_{w \in H \cap C_3} \phi(w)$

$\implies \phi_3^G = \frac{4}{13} \sum_{w \in H \cap C_3} \phi(w) = 0$ (since $H \cap C_3 = \phi$)

$\phi_4^G = \frac{4}{13} \sum_{w \in H \cap C_4} \phi(w)$

$\Longrightarrow\ \phi_4^G = \frac{4}{13}\sum_{w\in H\cap C_4}\phi(w) = 0$ (since $H\cap C_4 = \phi$)
$\phi_5^G = \frac{4}{4}\sum_{w\in H\cap C_5}\phi(w)$

$\phi_5^G = \frac{4}{4}\sum_{w\in H\cap C_5}\phi((1,9,4,12,7,2,10,5,13,8,3,11,6))$

$\Longrightarrow\ \phi_5^G = 1*\phi((1,9,4,12,...)) + 1*\phi((1,13,12,11,...)) + 1*\phi((1,2,3,...)) + 1*\phi((1,6,11,...)) = Z(13)_{13}^4 + Z(13)_{13}^6 + Z(13)_{13}^7 + Z(13)_{13}^9 = 20\ mod\ 53$,
since $Z(13)_{13} = 16$.

$\phi_6^G = \frac{4}{4}\sum_{w\in H\cap C_6}\phi(w)$

$\phi_6^G = \frac{4}{4}\sum_{w\in H\cap C_6}\phi((1,11,8,2,...)$

$\Longrightarrow\ \phi_6^G = 1*\phi((1,4,7,10,...)) + 1*\phi((1,12,10,8,...)) + 1*\phi((1,3,5,7,...)) + 1*\phi((1,11,8,5,...)) = Z(13)_{13}^8 + Z(13)_{13}^{12} + Z(13)_{13} + Z(13)_{13}^5 = 39\ mod\ 53$

$\phi_7^G = \frac{4}{4}\sum_{w\in H\cap C_7}\phi(w)$

$\phi_7^G = \frac{4}{4}\sum_{w\in H\cap C_7}\phi((1,10,6,2,11,...))$

$\Longrightarrow\ \phi_7^G = 1*\phi((1,10,6,2,11,...)) + 1*\phi((1,8,2,9,3,...)) + 1*\phi((1,7,13,6,12,...)) + 1*\phi((1,5,9,13,...)) = Z(13)_{13}^{11} + Z(13)_{13}^{10} + Z(13)_{13}^3 + Z(13)_{13}^2 = 46\ mod\ 53$.

so $\phi\uparrow_H^G = 4,0,0,0,20,39,46$ and we have verified that
$\chi_{.3}$ of H induces $\chi_{.5}$ of G.

Table 8.7: Character Table of G

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$\chi_{.1}$	1	1	1	1	1	1	1
$\chi_{.2}$	1	1	-1	-1	1	1	1
$\chi_{.3}$	1	-1	-I	I	1	1	1
$\chi_{.4}$	1	-1	I	-I	1	1	1
$\chi_{.5}$	4	0	0	0	Z_1	$Z_1\#2$	$Z_1\#4$
$\chi_{.6}$	4	0	0	0	$Z_1\#2$	$Z_1\#4$	Z_1
$\chi_{.7}$	4	0	0	0	$Z_1\#4$	Z_1	$Z_1\#2$

# denotes algebraic conjugation.
Z_1 is the primitive thirteen root of unity. I is the primitive fourth root of unity.

Table 8.8: Character Table of H

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	1	1	1	1	-1	-1	-1	-1
$\chi_{.3}$	1	1	1	-1	-1	I	-I	I	-I
$\chi_{.4}$	1	1	1	-1	-1	-I	I	-I	I
$\chi_{.5}$	1	-1	1	I	-I	Z_1	$Z_1\#3$	$-Z1$	$-Z1\#3$
$\chi_{.6}$	1	-1	1	-I	I	$Z_1\#3$	Z_1	$-Z_1\#3$	$-Z1$
$\chi_{.7}$	1	-1	1	-I	I	$-Z_1\#3$	$-Z_1$	$Z_1\#3$	Z_1
$\chi_{.8}$	1	-1	1	I	-I	$-Z_1$	$-Z_1\#3$	Z_1	$Z_1\#3$
$\chi_{.9}$	8	0	-1	0	0	0	0	0	0

# denotes algebraic conjugation.
Z_1 is the primitive thirteen root of unity.

Table 8.9: $\chi_{.5}$ of G

ϕ^G	Class	Size	Class Representative
4	C_1	1	Id(G)
0	C_2	13	(1, 3)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)
0	C_3	13	(1, 10, 3, 7)(4, 12, 13, 5)(6, 9, 11, 8)
0	C_4	13	(1, 7, 3, 10)(4, 5, 13, 12)(6, 8, 11, 9)
$Z_1 = 20$	C_5	4	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
$Z_1\#2 = 39$	C_6	4	(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)
$Z_1\#4 = 46$	C_7	4	(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)

Table 8.10: $\chi_{.3}$ of H

ϕ	Class	Size	Class Representative
1	D_1	1	Id(H)
$Z_1\#2 = 44$	D_2	1	(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)
$Z_1\#4 = 28$	D_3	1	(1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3, 11, 6)
$Z_1\#6 = 13$	D_4	1	(1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)
$Z_1\#8 = 42$	D_5	1	(1, 4, 7, 10, 13, 3, 6, 9, 12, 2, 5, 8, 11)
$Z_1\#10 = 46$	D_6	1	(1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7)
$Z_1\#12 = 10$	D_7	1	(1, 12, 10, 8, 6, 4, 2, 13, 11, 9, 7, 5, 3)
$Z_1 = 16$	D_8	1	(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)
$Z_1\#3 = 15$	D_9	1	(1, 7, 13, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8)
$Z_1\#5 = 24$	D_{10}	1	(1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)
$Z_1\#7 = 49$	D_{11}	1	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
$Z_1\#9 = 36$	D_{12}	1	(1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9)
$Z_1\#11 = 47$	D_{13}	1	(1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5)

Equivalencies from table to construct matrix: 1 = 1,
$Z_1 = 16$.

With induction taking place, now show the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} Z_{13}^7 & 0 & 0 & 0 \\ 0 & Z_{13}^4 & 0 & 0 \\ 0 & 0 & Z_{13}^6 & 0 \\ 0 & 0 & 0 & Z_{13}^9 \end{bmatrix}, \text{ and } A(yy) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Verifying the Monomial Representation

Since we have a linear character ϕ of the subgroup H of index 4 in G we let $G = Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4$ where the $t_i's$ are transversals of G acting on H.

That is G = He $\cup\, H(1,5,12,8)(2,10,11,3)(4,7,9,6) \cup H(1,12)(2,11)(3,10)(4,9)(5,8)(6,7) \cup H(1,8,12,5)(2,3,11,10)(4,6,9,7)$

Continuing the process in a 4x4 matrix:

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \phi(t_1xt_3^{-1}) & \phi(t_1xt_4^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \phi(t_2xt_3^{-1}) & \phi(t_2xt_4^{-1}) \\ \phi(t_3xt_1^{-1}) & \phi(t_3xt_2^{-1}) & \phi(t_3xt_3^{-1}) & \phi(t_3xt_4^{-1}) \\ \phi(t_4xt_1^{-1}) & \phi(t_4xt_2^{-1}) & \phi(t_4xt_3^{-1}) & \phi(t_4xt_4^{-1}) \end{bmatrix}$$

We will calculate all 16 elements of the matrix using the following calculation where $t_1 = e$, $t_2 = (1,5,12,8)(2,10,11,3)(4,7,9,6)$,
$t_3 = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$, $t_4 = (1,8,12,5)(2,3,11,10)(4,6,9,7)$, and $x = (1,2,3,4,5,6,7,8,9,10,11,12,13)$ with each t_i corresponding to each transversal respectively from the G listed before. We will start by calculating the multiplication of t_ixt_j^{-1} and list the resulting permutation rather than show the entire process. For example:
$\phi(t_1xt_1^{-1}) = \phi(e(1,2,3,4,5,6,7,8,9,10,11,12,13)e^{-1}) = \phi((1,2,3,4,5,6,7,8,9,10,11,12,13)$. We will go directly to what $\phi(t_1xt_1^{-1})$ is equal to. Since $(1,2,3,4,5,6,7,8,9,10,11,12,13) \in H$. We look back to our Table 5.4 note $\phi((1,2,3,4,5,6,7,8,9,10,11,12,13))$ is in class D_{11}=Z_1#7 $= 49$ *mod* 53 where $Z_1 = 16$. Therefore, the nonzero entry for *Row* 1 is 49. We proceed as follows:

Row 1:
$\phi(t_1xt_1^{-1}) = \phi((1,2,3,4,5,6,7,8,9,10,11,12,13)) = 49$
$\phi(t_1xt_2^{-1}) = \phi((1,3,6,4)(2,11,5,9)(7,12,13,8) = 0$
$\phi(t_1xt_3^{-1}) = \phi((1,11)(2,10)(3,9)(4,8)(5,7)(12,13)) = 0$
$\phi(t_1xt_4^{-1}) = \phi((1,10,3,7)(4,12,13,5)(6,9,11,8)) = 0$

Row 2:
$\phi(t_2xt_1^{-1}) = \phi((1,6,5,13)(2,11,4,8)(7,10,12,9)) = 0$
$\phi(t_2xt_2^{-1}) = \phi((1,9,4,12,7,2,10,5,13,8,3,11,6)) = 28$
$\phi(t_2xt_3^{-1}) = \phi((1,7,3,10)(4,5,13,12)(6,8,11,9)) = 0$
$\phi(t_2xt_4^{-1}) = \phi((1,4)(2,3)(5,13)(6,12)(7,11)(8,10)) = 0$

Row 3:
$\phi(t_3xt_1^{-1}) = \phi((1,13)(2,12)(3,11)(4,10)(5,9)(6,8)) = 0$
$\phi(t_3xt_2^{-1}) = \phi((1,13,8,9)(2,5,7,4)(3,10,6,12)) = 0$

$\phi(t_3xt_3^{-1}) = \phi((1,13,12,11,10,9,8,7,6,5,4,3,2)) = 13$
$\phi(t_3xt_4^{-1}) = \phi((1,13,5,6)(2,8,4,11)(7,9,12,10)) = 0$

Row 4:
$\phi(t_4xt_1^{-1}) = \phi((1,9,8,13)(2,4,7,5)(3,12,6,10)) = 0$
$\phi(t_4xt_2^{-1}) = \phi((1,7)(2,6)(3,5)(8,13)(9,12)(10,11)) = 0$
$\phi(t_4xt_3^{-1}) = \phi((1,4,6,3)(2,9,5,11)(7,8,13,12)) = 0$
$\phi(t_4xt_4^{-1}) = \phi((1,6,11,3,8,13,5,10,2,7,12,4,9)) = 36$

Each ϕ of H corresponded with a conjugacy class of either H or G. If the element is in a conjugacy class from H (seen in table 5.4) we write the value of ϕ for that class. Since our matrix was produced in cyclotomic field 53, we needed to produce an order 13 element in Z_{53}. In this case, 2 was chosen as the element of order 13. To complete this process, the matrix for yy should also be verified by repeating the process above.

Row 1:
$\phi(t_1yt_1^{-1}) = \phi((1,5,12,8)(2,10,11,3)(4,7,9,6)) = 0$
$\phi(t_1yt_2^{-1}) = \phi(e) = 1$
$\phi(t_1yt_3^{-1}) = \phi((1,8,12,5)(2,3,11,10)(4,6,9,7)) = 0$
$\phi(t_1yt_4^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$

Row 2:
$\phi(t_2yt_1^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$
$\phi(t_2yt_2^{-1}) = \phi((1,5,12,8)(2,10,11,3)(4,7,9,6)) = 0$
$\phi(t_2yt_3^{-1}) = \phi(e) = 1$
$\phi(t_2yt_4^{-1}) = \phi((1,8,12,5)(2,3,11,10)(4,6,9,7)) = 0$

Row 3:
$\phi(t_3yt_1^{-1}) = \phi((1,8,12,5)(2,3,11,10)(4,6,9,7)) = 0$
$\phi(t_3yt_2^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$
$\phi(t_3yt_3^{-1}) = \phi((1,5,12,8)(2,10,11,3)(4,7,9,6)) = 0$

$\phi(t_3yt_4^{-1}) = \phi(e) = 1$

Row 4:
$\phi(t_4yt_1^{-1}) = \phi(e) = 1$
$\phi(t_4yt_2^{-1}) = \phi((1,8,12,5)(2,3,11,10)(4,6,9,7)) = 0$
$\phi(t_4yt_3^{-1}) = \phi((1,12)(2,11)(3,10)(4,9)(5,8)(6,7)) = 0$
$\phi(t_4yt_4^{-1}) = \phi((1,5,12,8)(2,10,11,3)(4,7,9,6)) = 0$

Therefore,

$$A(xx) = \begin{bmatrix} 49 & 0 & 0 & 0 \\ 0 & 28 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 36 \end{bmatrix}, \text{ and } A(yy) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Since $|A(xx)| = 13$ and $|A(yy)| = 4$ with $|A(xx) \cdot A(yy)| = 4$, the order of our index, we know that $\langle A(xx) \text{ and } A(yy)\rangle$ is a faithful representation of G. We proceed and produce a permutation representation.

Constructing a Permutation Representation

We worked in Z_{13} on matrices of degree 4x4 which implies we are producing a $53^{*4} :_m 13 : 4$ progenitor permutation representation based on the monomial representation $13 : 4 = \langle A(xx), A(yy)\rangle$. We consider the matrix entries for $a_{i,j}$ for A(xx) and A(yy). We know there are 4 t's, one for each column of our matrix and because we have a semi-direct product in our progenitor then the elements of 13 : 4 act as automorphisms of $< t_1 > * < t_2 > * < t_3 > * < t_4 >$.

Now we represent xx and yy as permutations. To do so we observe

$$a_{i,j} = 1 \text{ if the automorphism takes } t_i \to t_j$$
$$a_{i,j} = n \text{ if the automorphism takes } t_i \to t_j^n.$$

Since 53^{*4} is a free product of four cyclic groups of order 52 we will construct a table with four t's of order 52 labeled from 1...208 found in table 5.5. Viewing the entries of A(xx)
$a_{11} = 49$, $a_{22} = 28$, $a_{33} = 13$, and $a_{44} = 36$.

Thus,

$t_1 \to t_1^{49}$,

$t_2 \to t_2^{28}$,

$t_3 \to t_3^{13}$.

$t_4 \to t_4^{36}$.

Proceeding on finding a permutation representative we must describe Table 5.5. The top number of the chart labels each element, the middle of the chart shows what the automorphism produces, and the bottom number shows what that produced element is numbered on the top of the chart. The same process is conducted for generators yy, but for Table 5.6 we will only show part of the process.

Table 8.11: Automorphisms of A(xx)

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_3	t_4	t_1^2	t_2^2	t_3^2	t_4^2	t_1^3	t_2^3	t_3^3	t_4^3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{49}	t_2^{28}	t_3^{13}	t_4^{36}	t_1^{45}	t_2^{3}	t_3^{26}	t_4^{19}	t_1^{41}	t_2^{31}	t_3^{39}	t_4^{2}
193	110	51	144	177	10	103	76	161	122	155	8

13	14	15	16	17	18	19	20	21	22	23	24
t_1^4	t_2^4	t_3^4	t_4^4	t_1^5	t_2^5	t_3^5	t_4^5	t_1^6	t_2^6	t_3^6	t_4^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{37}	t_2^{6}	t_3^{52}	t_4^{38}	t_1^{33}	t_2^{34}	t_3^{12}	t_4^{21}	t_1^{29}	t_2^{9}	t_3^{25}	t_4^{4}
145	22	207	152	129	134	47	84	113	34	99	16

25	26	27	28	29	30	31	32	33	34	35	36
t_1^7	t_2^7	t_3^7	t_4^7	t_1^8	t_2^8	t_3^8	t_4^8	t_1^9	t_2^9	t_3^9	t_4^9
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{25}	t_2^{37}	t_3^{38}	t_4^{40}	t_1^{21}	t_2^{12}	t_3^{51}	t_4^{23}	t_1^{17}	t_2^{40}	t_3^{11}	t_4^{6}
97	146	151	160	81	46	203	92	65	158	43	24

37	38	39	40	41	42	43	44	45	46	47	48
t_1^{10}	t_2^{10}	t_3^{10}	t_4^{10}	t_1^{11}	t_2^{11}	t_3^{11}	t_4^{11}	t_1^{12}	t_2^{12}	t_3^{12}	t_4^{12}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{13}	t_2^{15}	t_3^{24}	t_4^{42}	t_1^{9}	t_2^{43}	t_3^{37}	t_4^{25}	t_1^{5}	t_1^{18}	t_3^{50}	t_4^{8}
49	58	95	168	33	170	147	100	17	70	199	32

49	50	51	52	53	54	55	56	57	58	59	60
t_1^{13}	t_2^{13}	t_3^{13}	t_4^{13}	t_1^{14}	t_2^{14}	t_3^{14}	t_4^{14}	t_1^{15}	t_2^{15}	t_3^{15}	t_4^{15}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{1}	t_2^{46}	t_3^{10}	t_4^{44}	t_1^{50}	t_2^{21}	t_3^{23}	t_4^{27}	t_1^{46}	t_2^{49}	t_3^{36}	t_4^{10}
1	182	39	176	197	82	91	108	181	194	143	40

61	62	63	64	65	66	67	68	69	70	71	72
t_1^{16}	t_2^{16}	t_3^{16}	t_4^{16}	t_1^{17}	t_2^{17}	t_3^{17}	t_4^{17}	t_1^{18}	t_2^{18}	t_3^{18}	t_4^{18}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{42}	t_2^{24}	t_3^{49}	t_4^{46}	t_1^{38}	t_2^{52}	t_3^{9}	t_4^{29}	t_1^{34}	t_2^{27}	t_3^{22}	t_4^{12}
167	94	195	184	149	206	35	116	133	106	87	48

73	74	75	76	77	78	79	80	81	82	83	84
t_1^{19}	t_2^{19}	t_3^{19}	t_4^{19}	t_1^{20}	t_2^{20}	t_3^{20}	t_4^{20}	t_1^{21}	t_2^{21}	t_3^{21}	t_4^{21}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{30}	t_2^{2}	t_3^{35}	t_4^{48}	t_1^{26}	t_2^{30}	t_3^{48}	t_4^{31}	t_1^{22}	t_2^{5}	t_3^{8}	t_4^{14}
117	6	139	192	101	118	191	124	85	18	31	56

85	86	87	88	89	90	91	92	93	94	95	96
t_1^{22}	t_2^{22}	t_3^{22}	t_4^{22}	t_1^{23}	t_2^{23}	t_3^{23}	t_4^{23}	t_1^{24}	t_2^{24}	t_3^{24}	t_4^{24}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{18}	t_2^{33}	t_3^{21}	t_4^{50}	t_1^{14}	t_2^{8}	t_3^{34}	t_4^{33}	t_1^{10}	t_2^{36}	t_3^{47}	t_4^{16}
69	130	83	200	53	30	135	132	37	142	187	64

97	98	99	100	101	102	103	104	105	106	107	108
t_1^{25}	t_2^{25}	t_3^{25}	t_4^{25}	t_1^{26}	t_2^{26}	t_3^{26}	t_4^{26}	t_1^{27}	t_2^{27}	t_3^{27}	t_4^{27}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{6}	t_2^{11}	t_3^{7}	t_4^{52}	t_1^{2}	t_2^{39}	t_3^{20}	t_4^{35}	t_1^{51}	t_2^{14}	t_3^{33}	t_4^{18}
21	42	27	208	5	154	79	140	201	54	131	72

109	110	111	112	113	114	115	116	117	118	119	120
t_1^{28}	t_2^{28}	t_3^{28}	t_4^{28}	t_1^{29}	t_2^{29}	t_3^{29}	t_4^{29}	t_1^{30}	t_2^{30}	t_3^{30}	t_4^{30}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{47}	t_2^{42}	t_3^{46}	t_4^{1}	t_1^{43}	t_2^{17}	t_3^{6}	t_4^{37}	t_1^{39}	t_2^{45}	t_3^{19}	t_4^{20}
185	166	183	4	169	66	23	148	153	178	75	80

121	122	123	124	125	126	127	128	129	130	131	132
t_1^{31}	t_2^{31}	t_3^{31}	t_4^{31}	t_1^{32}	t_2^{32}	t_3^{32}	t_4^{32}	t_1^{33}	t_2^{33}	t_3^{33}	t_4^{33}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{35}	t_2^{20}	t_3^{32}	t_4^{3}	t_1^{31}	t_2^{48}	t_3^{45}	t_4^{39}	t_1^{27}	t_2^{23}	t_3^{5}	t_4^{22}
137	78	127	12	121	190	179	156	105	90	19	88

133	134	135	136	137	138	139	140	141	142	143	144
t_1^{34}	t_2^{34}	t_3^{34}	t_4^{34}	t_1^{35}	t_2^{35}	t_3^{35}	t_4^{35}	t_1^{36}	t_2^{36}	t_3^{36}	t_4^{36}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{23}	t_2^{51}	t_3^{18}	t_4^{5}	t_1^{19}	t_2^{26}	t_3^{31}	t_4^{41}	t_1^{15}	t_2^{2}	t_3^{44}	t_4^{24}
89	202	71	20	73	102	123	164	57	2	175	96

145	146	147	148	149	150	151	52	153	154	155	156
t_1^{37}	t_2^{37}	t_3^{37}	t_4^{37}	t_1^{38}	t_2^{38}	t_3^{38}	t_4^{38}	t_1^{39}	t_2^{39}	t_3^{39}	t_4^{39}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{11}	t_2^{29}	t_3^{4}	t_4^{7}	t_1^{7}	t_2^{4}	t_3^{17}	t_4^{43}	t_1^{3}	t_2^{32}	t_3^{30}	t_4^{26}
41	114	15	28	25	14	67	172	9	126	119	104

157	158	159	160	161	162	163	164	165	166	167	168
t_1^{40}	t_2^{40}	t_3^{40}	t_4^{40}	t_1^{41}	t_2^{41}	t_3^{41}	t_4^{41}	t_1^{42}	t_2^{42}	t_3^{42}	t_4^{42}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{52}	t_2^{7}	t_3^{43}	t_4^{9}	t_1^{48}	t_2^{35}	t_3^{3}	t_4^{45}	t_1^{44}	t_2^{10}	t_3^{16}	t_4^{28}
205	26	171	36	189	138	11	180	173	38	63	112

169	170	171	172	173	174	175	176	177	178	179	180
t_1^{43}	t_2^{43}	t_3^{43}	t_4^{43}	t_1^{44}	t_2^{44}	t_3^{44}	t_4^{44}	t_1^{45}	t_2^{45}	t_3^{45}	t_4^{45}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{40}	t_2^{38}	t_3^{29}	t_4^{11}	t_1^{36}	t_2^{13}	t_3^{42}	t_4^{47}	t_1^{32}	t_2^{41}	t_3^{2}	t_4^{30}
157	150	115	44	141	50	167	188	125	162	7	120

181	182	183	184	185	186	187	188	189	190	191	192
t_1^{46}	t_2^{46}	t_3^{46}	t_4^{46}	t_1^{47}	t_2^{47}	t_3^{47}	t_4^{47}	t_1^{48}	t_2^{48}	t_3^{48}	t_4^{48}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{28}	t_2^{16}	t_3^{15}	t_4^{13}	t_1^{24}	t_2^{44}	t_3^{28}	t_4^{49}	t_1^{20}	t_2^{19}	t_3^{41}	t_4^{32}
109	62	59	52	93	174	111	196	77	74	163	128

193	194	195	196	197	198	199	200	201	202	203	204
t_1^{49}	t_2^{49}	t_3^{49}	t_4^{49}	t_1^{50}	t_2^{50}	t_3^{50}	t_4^{50}	t_1^{51}	t_2^{51}	t_3^{51}	t_4^{51}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{16}	t_2^{47}	t_3^{1}	t_4^{15}	t_1^{12}	t_2^{22}	t_3^{14}	t_4^{51}	t_1^{8}	t_2^{50}	t_3^{27}	t_4^{34}
61	186	3	60	45	86	55	204	29	198	107	136

205	206	207	208
t_1^{52}	t_2^{52}	t_3^{52}	t_4^{52}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^{4}	t_2^{25}	t_3^{40}	t_4^{17}
13	98	159	68

Table 8.12: Automorphisms of A(yy)

1	2	3	...	...	...	206	207	208
t_1	t_2	t_3	...	...	...	t_8^{16}	t_9^{16}	t_{10}^{16}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_3	t_4	...	...	...	t_3^{52}	t_4^{52}	t_1^{52}
2	3	4	...	...	...	207	208	205

From Table 8.11 and 8.12, we can construct permutations for $A(xx)$ and $A(yy)$ by using our labels from each automorphism. Consider element #1 be denoted as t_1 from Table 8.11. This produces the permutation $(1, 193, 61, 165, 173, 141, 57,$ $181, 109, 185, 93, 37, 49)$. If we follow each element and its corresponding automorphism number labeling and repeating the process for xx, we produce the following permutations:

$A(xx) = (1, 193, 61, 165, 173, 141, 57, 181, 109, 185, 93, 37, 49),$
$(2, 110, 166, 38, 58, 194, 186, 174, 50, 182, 62, 94, 142),$
$(3, 51, 39, 95, 187, 111, 183, 59, 143, 175, 167, 63, 195),$
$(4, 144, 96, 64, 184, 52, 176, 188, 196, 60, 40, 168, 112),$
$(5, 177, 125, 121, 137, 73, 117, 153, 9, 161, 189, 77, 101),$
$(6, 10, 122, 78, 118, 178, 162, 138, 102, 154, 126, 190, 74),$
$(7, 103, 79, 191, 163, 11, 155, 119, 75, 139, 123, 127, 179),$
$(8, 76, 192, 128, 156, 104, 140, 164, 180, 120, 80, 124, 12),$
$(13, 145, 41, 33, 65, 149, 25, 97, 21, 113, 169, 157, 205),$
$(14, 22, 34, 158, 26, 146, 114, 66, 206, 98, 42, 170, 150),$
$(15, 207, 159, 171, 115, 23, 99, 27, 151, 67, 35, 43, 147),$
$(16, 152, 172, 44, 100, 208, 68, 116, 148, 28, 160, 36, 24),$

$(17, 129, 105, 201, 29, 81, 85, 69, 133, 89, 53, 197, 45)$,
$(18, 134, 202, 198, 86, 130, 90, 30, 46, 70, 106, 54, 82)$,
$(19, 47, 199, 55, 91, 135, 71, 87, 83, 31, 203, 107, 131)$,
$(20, 84, 56, 108, 72, 48, 32, 92, 132, 88, 200, 204, 136)$

Likewise for yy.

$A(yy) = (1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12), (13, 14, 15, 16)$,
$(17, 18, 19, 20), (21, 22, 23, 24), (25, 26, 27, 28), (29, 30, 31, 32)$,
$(33, 34, 35, 36), (37, 38, 39, 40), (41, 42, 43, 44), (45, 46, 47, 48)$,
$(49, 50, 51, 52), (53, 54, 55, 56), (57, 58, 59, 60), (61, 62, 63, 64)$,
$(65, 66, 67, 68), (69, 70, 71, 72), (73, 74, 75, 76), (77, 78, 79, 80)$,
$(81, 82, 83, 84), (85, 86, 87, 88), (89, 90, 91, 92), (93, 94, 95, 96)$,
$(97, 98, 99, 100), (101, 102, 103, 104), (105, 106, 107, 108), (109, 110, 111, 112)$,
$(113, 114, 115, 116), (117, 118, 119, 120), (121, 122, 123, 124), (125, 126, 127, 128)$,
$(129, 130, 131, 132), (133, 134, 135, 136), (137, 138, 139, 140), (141, 142, 143, 144)$,
$(145, 146, 147, 148), (149, 150, 151, 152), (153, 154, 155, 156), (157, 158, 159, 160)$,
$(161, 162, 163, 164), (165, 166, 167, 168), (169, 170, 171, 172), (173, 174, 175, 176)$,
$(177, 178, 179, 180), (181, 182, 183, 184), (185, 186, 187, 188), (189, 190, 191, 192)$,
$(193, 194, 195, 196), (197, 198, 199, 200), (201, 2012, 203, 204), (205, 206, 207, 208)$
therefore, we have constructed a permutation representation from our matrices.

Creating a Presentation of the Progenitor

To construct a presentation for the progenitor we must choose a t to normalize from our four choices $< t_1 > * < t_2 > * < t_3 > * < t_4 >$ Let $t \sim t_1$ and we must find permutations which normalizes $< t_1 >$, or fixes the following set
$\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, t_1^{13}, t_1^{14}, t_1^{15}, t_1^{16}, t_1^{17}\}$. This is a defining characteristic of a monomial progenitors. Monomial progenitors fix a set of t's while permutation progenitors fix only one specific t_i.
Using Magma, we were able to find the normaliser stabiliser.

```
Normaliser:=Stabiliser(N,(1,5,9,13,17,21,25,29,33,37,41,45,
49,53,57,61, 65,69,73,77,81,85,89,93,97,101,105,109,113,
117,121,125,129,133,137,141,145,149,153,157,161,165,169,
173,177,181,185,189,193,197,201,205));

norm = (1,37,185,181,141,165,193,49,93,109,57,173,61)
(2,94,182,174,194,38,110,142,62,50,186,58,166)(3, 63,175,59,
111,95,51,195,167,143,183,187, 39)(4,168,60,188,52,64,144,112,
40,196,176,184,96)(5,77,161,153,73,121,177,101,189,9,117,137,
125)(6,190,154,138,178,78,10,74,126,102,162,118,122)(7,127,
139,119,11,191,103,179,123,75,155,163,79)(8,124,120,164,104,
128,76,12,80,180,140,156,192)(13,157,113,97,149,33,145, 205,
169,21,25,65,41)(14,170,98,66,146,158,22,150,42,206,114,26,
34)(15,43,67,27,23,171,207,147,35,151,99,115,159)(16,36, 28,
116,208,44,152,24,160,148,68,100,172)(17,197,89,69,81,201,
129,45,53,133,85,29,105)(18,54,70,30,130,198,134,82,106,46,
90,86,202)(19,107,31,87,135,55,47,131,203,83,71,91,199)(20,
204,88,92,48,108,84,136,200,132,32,72,56)
```

For a presentation we must convert this permutation into words which we find to be: norm $= x^{-2}$.
We look at the permutation $xx^{-2} = (1, 37, 185, ...)$, 1 goes to 37 so we check the label in our Table 5.5. If the element is 37, then the automorphism is t_1^{10}

$$\text{norm} \to t^{x^{-2}} = t_1^{10}.$$

So, a presentation for the progenitor is $53^{*4} :_m 13 : 4 = G < x, y, t >:= Group < x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^2, t^{53}, t^{x^{-2}} = t^{10} >$.
Now we check if our progenitor is correct. We apply Grindstaff's Lemma. Our symmetric generators are t_1, t_2, t_3, and t_4. We want to add to the above presentation that all $ti's$ commute; that is, (t_1, t_2), (t_1, t_3), and (t_1, t_4). Now $t_2 = t^y$, $t_3 = t^{y^2}$, and $t_4 = t^{y^3}$.
So we check if
$53^{*4} :_m 13 : 4 = G < x, y, t >:= Group < x, y, t | y^4, y^{-2}x^{-1}y^2x^{-1}, y^{-1}x^{-3}yx^2, t^{53},$
$t^{x^{-2}} = t^{10} >$ factored by (t, t^y), (t, t^{y^2}), and (t, t^{y^3}) is the group $53^4 :_m (13 : 4)$ of order $53^4 \times (13 \times 4)$. Next, we find finite homomorphic images of the progenitor $53^{*4} :_m (13 : 4)$. Thus, we factor the progenitor by additional relations.

Chapter 9

Computing Large Finite Homomorphic Images

9.1 Finding Homomorphic Image

The program $MAGMA$ is used to compute large number of finite groups. But if we have a group with a large number of permutations representation then we use two different methods in MAGMA to calculate the order of G and its composition factor. The loops that were used through this process are listed below. The following loops helped to compute the order and composition factor of a large finite group.

Method(1)

One of the groups we used to compute the order and its composition factor is:

```
>G<v,w,x,y,z,t>:=Group<v,w,x,y,z,t|v^2,w^3,x^2,y^2,z^2,
(w^-1*v)^2, w^-1*x*w*y,w*x*w^-1*z,v*x*v*y,(x*y)^2,
t^2,
(t,v * x * y * z * w^-1),(t,x),(v*t)^10,(v * w * y * z*t^v)^5,
(v * w * y * z*t)^6,(w*y*t)^0,(w*y*t^y)^4>;
```

Now we factor G by the element y and t, we could have used any of the generators. In this case we used on two. With that being said, now we find the index of G and H.

```
>H:=sub<G|y,t>;
>Index(G,H);
153600
```

The Index of G and H is 153600, which is a number that can be compute using Magma. now we do the coset action of group G and H. Then we check the kernel. To get an interesting Image we need to get kernel to be 1 otherwise is not helpful.

```
>f,G1,k:=CosetAction(G,H);
>#k;
1
```

In this case kernel is 1 so we proceed by checking the composition factor of the group, G.

```
>CompositionFactors(G1);
G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

Therefore, $G \cong 2^8 :^{\bullet} (A_5 : (S_5 : 2))$

Method(2)

Example1 For the following group we used a different method to calculate the order of G and to find the composition factor. In this case we were able to used this method since the index of the group did not exceed the order of the index, which is of order 7^{10}

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^3,z^2,w^2, (y^-1*x)^2,
 y^-1*z*y*w,(x*z)^2,(z*w)^2, y*z*y^-1*z*w,
 t^2,
 (t,x*y),(x * y^-1 * w*t)^6,
(y*t^(x*w^-1))^4,(x*w*t)^4,(x*w*t^y)^10>;
V:=CosetSpace(G,sub<G|x,y,z,w>: CosetLimit:=7^10,
 Hard:=true, Print:=1);
INDEX = 9360000
G:=CosetImage(V);
CompositionFactors(G);
    G
    |  Cyclic(2)
    *
    |  A(1, 25)                    = L(2, 25)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Alternating(5).
    *
    |  Cyclic(2)
    1
```

Therefore, $G \cong 2 : (A_5 \times (S_5 : PGL2(25)))$

Example2

Since the order of G is 18720000 exceeding the number of permutation representation that MAGMA can calculate. If we want to compute the order of the group and its composition factor, the we must use the following loop in MAGMA:

```
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,y^-1*x^-3*yx^-2,
t^2,
(t,x^-1 * y^-1 * x), (y * x^2 * y*t)^4,(x^2*t)^5>;
V:=CosetSpace(G,sub<G|x,y>: CosetLimit:=7^10,
Hard:=true, Print:=1); G:=CosetImage(V); CompositionFactors(G);
INDEX = 360000 (a=360000 r=477 h=441134 n=441134; l=886
c=1.19; m=360001 t=441133)
    G
    |  C(2, 5)                     = S(4, 5)
    *
```

```
 |  Cyclic(2)
 *
 |  Cyclic(2)
 1
```

Therefore $G \cong 4^{\bullet}S_4(5)$.

9.2 Factoring $4^{\bullet}S(4,5)$ over $(13:4)$ by $Z(G)$

From a previous example we were able to obtained an important finite homomorphic image, $4^{bullet}S_4(5)$, using the famous lemma. (More details and information about this group will be found in Chapter 2.) In this section we are going to factor G by the center, which is of order 4. To factor $4^{\bullet}S_4(5)$ over (13 : 4 by the center or also known as $Z(G)$, we are going to use a similar process we used on the previous example. Let $G = \frac{(2^{*13}:13):4}{(yx^2yt_1)^4,(x^2t_1)^5} \cong 4^{\bullet}S_4(5)$. Since we want G to be factored by the first order relations $(yx^2yt_1)^4, (x^2t_1)^5$, and the center we must run the following loops in $MAGMA$ so we can get our desired.

```
>G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,y^-1*x^-3*y*x^-2,
t^2,(t,x^-1 * y^-1 * x), (y * x^2 * y*t)^4,(x^2*t)^5>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> #G1;
18720000
> CompositionFactors(G1);
 G
 | C(2, 5)  = S(4, 5)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 1
> C:=Center(G1);
> #C;
4
> Order(C.1);
4
> D:=C.1;
> Order(D);
4
```

Thus, the order of the center of G is 4.

Since we know the order of the center, we proceed by converting the $Z(G)$ in terms of words, to do so, we are going to use the WordGroup loop.

```
> W,phi:=WordGroup(G1);
> rho:=InverseWordMap(G1);
> D@rho;
> gg:=function(W)
w4 := W.1 * W.2; w5 := w4 * W.3; w6 := w5 * W.1;
end function;
> gg(G);
x * y * t * x * t * x^2 * t * y * x^-1 * t * y * t *
y^-1 * t * y^-1 * t * x^2 *t * x * t
> #k;
1
```

Hence, we have found the center of G and converted into words,

$Z(G) =< xytxtx^2tyx^{-1}tyty^{-1}ty^{-1}tx^2txt >$. As of now, we factor the finite presentation, G, by the relations and the center,$Z(G)$. To verify that we have factored G by the center we run the following loops in MAGMA:

```
> G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,y^-1*x^-3*yx^-2,
t^2,
(t,x^-1 * y^-1 * x), (y * x^2 * y*t)^4,
(x^2*t)^5,x * y * t * x * t * x^2 * t * y * x^-1 * t * y * t *
 y^-1 * t * y^-1 * t * x^2 * t * x * t>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
>CompositionFactors(G1);
 G
    |  C(2, 5)                        = S(4, 5)
    1
```

In Conclusion we have showed that G is factored by the relations $(yx^2yt_1)^4, (x^2t_1)^5$ and by the center $Z(G) =< xytxtx^2tyx^{-1}tyty^{-1}ty^{-1}tx^2txt >$.

9.2.1 Construction of $S(4,5)$ over 13:4

Now that we know the center of G we proceed on factoring G by the relations and the center to obtained $G \cong \frac{2^{*13}:(13:4)}{(yx^2yt_1)^4,(x^2t_1)^5,xytxtx^2tyx^{-1}tyty^{-1}ty^{-1}tx^2txt} \cong S_4(5)$. Note $N = (13:4)$, where $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13)$ and $y \sim 1,8,12,5)(2,3,11,10)(4,6,9,7)$. Let $t \sim t_1$.

```
> G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,x^-1 * y^-1 * x), (y * x^2 * y*t)^4,
(x^2*t)^5,x * y * t * x * t * x^2 * t * y * x^-1 * t * y * t *
 y^-1 * t * y^-1 * t * x^2 * t * x * t>;
> Index(G,sub<G|x,y>);
90000
f,G1,k:=CosetAction(G,sub<G|x,y>);
> #DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
1767
```

We want to find the index of N in G. To do so, we regularly perform a manual double coset enumeration of G over N. But noticed from the information listed above, by cause of a large number of double cosets of G we are going to stop here.

Chapter 10

Tables of Homomorphic Images

10.1 Homomorphic Images of $2^{*13} : 13$

```
|N|=13
G<x,t>:=Group<x,t|x^-13,
t^2,
(x^-1*t)^b,(x*t*t^x)^c,(x^-2*t)^d,(x^2*t*t^x)^e,
(x^-3*t)^f,(x^3*t*t^x)^g,(x^-4*t)^h,(x^4*t*t^x)^i,
(x^-5*t)^j,(x^5*t*t^x)^k,(x^-6*t)^l,(x^6*t*t^x)^m>;
```

Table 10.1: Some Finite Images of Progenitor $2^{*13} : 13$

$\#G$	b	c	...	i	j	k	l	m	$Isomorphism\ Type$
1092	0	0	...	0	0	0	7	2	$PSL_2(13)$
2184	0	0	...	0	3	0	6	4	$PSL_2(25)$
7800	0	0	...	2	6	0	0	0	$2^{\bullet}PSL_2(13)$
11232	0	0	...	0	0	8	8	2	$PGL_3(3)$
15600	0	0	...	2	0	4	0	0	$PSL(25) : 2$
19656	0	0	...	0	4	0	0	2	$PSL_2(27) : 2$
124800	0	0	...	0	6	0	10	2	$U_3(4) : 2$
246480	0	0	...	0	0	0	3	4	$PSL_2(79)$
492960	0	0	...	0	0	2	0	4	$2^{\bullet}\ PSL_2(79)$
4094064	0	0	...	0	3	0	8	3	$2\times\ PSL_2(79)$
12130560	10	2	d=8, f=8...	0	0	0	0	0	

10.2 Homomorphic Images of $2^{*13} : (13 : 2)$

```
|N|=26
 G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-13,
 t^2,
 (t,b * x^2),
((x^6)*t)^c,(y*t^(x^-1)*t^(x^2))^d,
((x^5)*t)^e,(y*t*t^(x^3))^f,
((x^4)*t)^g,(y*t^x*t^(x^2))^h,
((x^3)*t)^i,(y*t^(x^2)*t)^j,
((x^2)*t)^k,(y*t^(x^3)*t)^l,
(x*t)^m, (y*t^(x^4)*t)^n,
(y*t^(x^5))^o,(y*t^(x^5)*t)^p,
 (y*t^(x^4))^q,(x*t*t^x)^r,
(y*t^(x^3))^s,((x^2)*t*t^x)^u,
 (y*t^(x^2))^v,((x^3)*t*t^x)^w,
(y*t^x)^z,((x^4)*t*t^x)^a1,
(y*t)^a2,((x^5)*t*t^x)^a3,
(y*t^(x^-1))^a4,((x^6)*t*t^x)^a5>;
```

Table 10.2: Some Finite Images of Progenitor $2^{*13} : (13 : 2)$

$\#G$	c	...	q	u	v	w	z	$a1$	$a2$	$a3$	$a4$	$a5$	$Isomorphism\ Type$
26	0	...	0	0	0	0	0	0	0	0	1	0	Empty
985920	0	...	0	0	0	0	0	0	0	2	0	4	4: PSL(2,79)
52	0	...	0	0	0	0	0	0	2	0	0	0	C_2
297648	0	...	0	0	0	0	0	0	4	2	0	0	$2^{\bullet}PGL_2(53)$
31200	0	...	0	0	0	0	0	2	0	4	10	0	$2^{\bullet}PGL_2(25)$
22464	0	...	4	3	4	0	0	0	0	0	0	0	$2^{\bullet}PGL_3(3)$
39312	0	...	0	0	0	0	0	3	0	0	0	2	$2 : PGL_2(27)$
19656	0	...	0	3	3	0	0	0	0	0	0	0	$PGL_2(27)$
2184	0	...	0	0	0	0	3	0	0	2	10	6	$2^{\bullet}PSL_2(13)$
11232	0	...	0	0	0	0	0	3	3	0	0	4	$2 \times PSL_2(13)$

10.3 Homomorphic Images of $2^{*13} : (13 : 4)$

```
|N|=52
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,y^-1*x^-3*y*x^-2,
t^2,
(t,y^x),
((x*y)^2*t^(x^-4))^c,((x*y)^2*t)^d,
((x*y)^2*t^x)^e,((x*y)^2*t^(x^6))^f,
((x*y)*t^(x^-4))^g,((x*y)*t)^h,
((x*y)*t^x)^i,((x*y)*t^(x^6))^j,
((y^-1*x^-1)*t^(x^-4))^k,
((y^-1*x^-1)*t)^l,((y^-1*x^-1)*t^x)^m,
((y^-1*x^-1)*t^(x^6))^n,
(x*t)^o,(x^2*t)^p,(x^4*t)^q>;
```

Table 10.3: Some Finite Images of Progenitor $2^{*13} : (13 : 4)$

$\#G$	c	d	...	l	m	n	o	p	q	*Isomorphism Type*
104	0	0	...	0	0	0	0	0	2	C_2
1352	0	0	...	0	0	0	4	0	0	$13^2 : (4 \times 2)$
58240	0	0	...	0	0	5	0	5	0	$2 \times S_z(8)$
29120	0	0	...	0	5	0	5	0	7	$S_z(8)$

10.4 Homomorphic Images of $2^{*13} : (13 : S_3)$

```
|N|=78
G<x,y,t>:=Group<x,y,t|y^6,y^-1*x^3*y*x,x^-4*y^-1*x*y,
t^2,
(t,x *y*x*y^2),(t,x^-1*y^-1*x),(y*x^2*y^2*t^(x^2))^a,
(y*x^2*y^2*t)^b,(y*x^2*y^2*t^x)^c,((x*y)^2*t^(x^2))^d,
((x*y)^2*t)^e,((x*y)^2*t^x)^f,((y^-1*x^-1)^2*t^(x^2))^g,
((y^-1*x^-1)^2*t)^h,((y^-1*x^-1)^2*t^x)^i,(x*y*t^(x^2))^j,
(x*y*t)^k,(x*y*t^x)^l,(y^-1*x^-1*t^(x^2))^m,
(y^-1*x^-1*t)^n,(y^-1*x^-1*t^x)^o,(x*t)^p,(x^2*t)^q>;
```

Table 10.4: Some Finite Images of Progenitor $2^{*13} : (13 : S_3)$

$\#G$	a	b	...	l	m	n	o	p	q	*Isomorphism Type*
156	0	0	...	0	0	0	0	0	2	C_2
638976	0	0	...	0	0	0	0	0	4	$2^{13} : (13 : S_3)$
26364	0	0	...	0	0	0	6	0	6	$13^3 :^{\bullet} S_3 : 2$

10.5 Homomorphic Images of $2^{*30} : (2^{\bullet}3 : 5)$

```
|N|=30
G<x,t>:=Group<x,t|x^30,
t^3,
(x^15*t)^a,(x^10*t)^b,(x^-10*t)^c,
(x^6*t)^d,(x^12*t)^e,(x^-12*t)^f,
(x^-6*t)^g,(x^5*t)^h,(x^-5*t)^i,
(x^3*t)^j,(x^9*t)^k,(x^-9*t)^l,
(x^-3*t)^m,(x^2*t)^n,(x^4*t)^o,
(x^8*t)^p,(x^14*t)^q,(x^-14*t)^r,
(x^-8*t)^s,(x^-4*t)^u,
(x^-2*t)^v,(x*t)^w,(x^7*t)^y,
(x^11*t)^z,(x^13*t)^aa,
(x^-13*t)^bb,(x^-11*t)^cc,
(x^-7*t)^dd,(x^-1*t)^ee>;
```

Table 10.5: Some Finite Images of Progenitor $2^{*30} : (2^{\bullet}3 : 5)$

$\#G$	a	b	...	z	aa	bb	cc	dd	ee	$Isomorphism\ Type$
180	0	0	...	0	0	0	0	3	2	A_5
60	0	0	...	0	0	0	2	3	2	A_5
3960	0	0	...	0	0	0	2	4	2	$3 \times PGL_2(11)$
241920	0	0	...	0	0	2	3	4	5	$4 : (PSL_3(4) \times 3)$
14880	0	0	...	0	3	0	4	5	2	$PSL_2(31)$
12180	0	0	...	0	3	0	5	7	2	$PSL_2(29)$

10.6 Homomorphic Images of $2^{*12} : S_4$

```
|N|=24
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^3,z^2,w^2,(y^-1*x)^2,y^-1
*z*y*w, (x*z)^2,(z*w)^2, y*z*y^-1*z*w,
t^2,
(t,x*y),
(x*y^-1*w*t)^m1,(z*t^y)^a,(z*t)^b,(x*t^(y^z))^c,(x*t^y)^d,
(x*t)^e,(x*t^w)^f,(y*t)^g,(y*t)^h,(y*t^w)^i,(y*t^(x*w^-1))^j,
(x*w*t)^k,(x*w*t^y)^l,(x*w*t^w)^m>;
```

Table 10.6: Some Finite Images of Progenitor $2^{*12} : S_4$

$\#G$	a	b	...	i	j	k	l	m	$m1$	$Isomorphism\ Type$
1008	0	0	...	0	0	0	3	0	3	$6 \times PSL_2(7)$
24360	0	0	...	0	0	0	7	0	2	$PGL_2(29)$
18579460	0	0	...	0	0	0	3	0	4	$2^{8^\bullet}3 : (S_6 \times PSL_2(7))$
720	0	0	...	0	0	0	3	4	10	$2^\bullet A_6$
11520	0	0	...	0	0	0	3	5	8	$2^4 : S_6$
336	0	0	...	0	0	0	3	7	3	$2 : PSL_2(7)$
2688	0	0	...	0	0	0	3	7	4	$2^4 : PSL_2(7)$
11232	0	0	...	0	0	0	4	8	3	$2^\bullet PSL_3(3)$
30720	0	0	...	0	0	0	5	5	3	$2^8 : (S_5)$
2016	0	0	...	0	0	0	6	0	2	$6 : PGL_2(7)$
3326400	0	0	...	0	0	0	6	5	3	$A_7 :^\bullet PGL_2(11)$
4896	0	0	...	0	0	0	9	4	3	$2^\bullet PSL_2(17)$
3960	0	0	...	0	2	0	5	0	6	$3 : PGL_2(11)$
672	0	0	...	0	8	0	2	0	8	$2^\bullet PGL_2(7)$
3420	0	0	...	0	9	0	2	0	9	$PSL_2(19)$
2184	0	0	...	0	2	0	7	7	6	$PGL_2(13)$
40320	0	0	...	0	3	0	5	5	8	$2 \times PSL_3(4)$
1440	0	0	...	0	3	0	6	4	4	$4^\bullet(3 : S_5)$
178920	0	0	...	0	3	0	7	5	4	$PSL_2(71)$
21504	0	0	...	0	4	0	3	7	6	$2^{7^\bullet} PSL_2(7)$
336	0	0	...	0	4	0	3	7	9	$2 : PSL_2(7)$
1062720	0	0	...	0	4	0	8	4	6	$2 : PGL_2(81)$
7680	0	0	...	0	4	4	6	0	6	$2^6 : (S_5)$
224640000	0	0	...	0	4	4	10	0	6	$2 : (A_5 \times (A_5^\bullet 2 : PGL_2(25)))$

```
|N|=24
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^3,z^2,w^2,
(y^-1*x)^2, y^-1*z*y*w, (x*z)^2,(z*w)^2, y*z*y^-1*z*w,
t^2,
(t,x*y),(x * y^-1 * w*t)^j,(x*w*t)^p1,
(x*t^y)^p2,(x*t)^p3,(y*t)^p4>;
```

Table 10.7: More Finite Images of Progenitor $2^{*12} : S_4$

$\#G$	$p1$	$p2$	$p3$	$p4$	j	$Isomorphism\ Type$
240	0	0	0	2	4	$2 \times S_5$
2184	0	0	0	2	7	$PGL_2(13)$
120	0	0	2	0	3	S_5
14400	0	0	2	0	5	$2 :^{\bullet} (A_5 \times S_5)$
15360	0	0	6	8	2	$2^7 :^{\bullet} S_5$
117600	0	0	7	8	2	$PGL_2(49)$
1344	0	0	0	3	3	$2^3 : PSL_2(7)$
4896	4	0	0	0	3	$2 :^{\bullet} PSL_2(17)$
28800	4	0	0	4	6	$2 :^{\bullet} (A_5 \times (S_5 \times 2))$
31457280	4	0	0	4	8	$2^{18} : S_5$
61440	5	0	0	4	4	$2^9 : S_5$
11796480	4	10	0	5	5	$2^{15} :^{\bullet} A_6$
2520	5	0	0	5	3	A_7
1920	5	0	0	6	3	$2^4 : S_5$
483840	10	0	0	10	2	$6 : (2 \times PGL_3(4))$
5040	5	4	3	0	5	$2^{\bullet} A_7$
30720	5	0	3	0	8	$2^{8^{\bullet}} S_5$
302400	5	0	3	0	10	$S_5 : A_7$
80640	7	0	3	0	5	$4 : PSL_3(4)$

```
|N|=48
G<v,w,x,y,z,t>:=Group<v,w,x,y,z,t|v^2,w^3,x^2,y^2,z^2,
(w^-1*v)^2, w^-1*x*w*y, w*x*w^-1*z,v*x*v*y,(x*y)^2,
t^2,
(t,v * x * y * z * w^-1),(t,x),(v*t)^l,
(w*t^y)^m1,(v*t^(x*w^2))^m2,
(v * w * y * z*t)^m3>;
```

10.7 Homomorphic Images of $2^{*12} : (S_4 \times 2)$

Table 10.8: $2^{*12} : (S_4 \times 2)$

$\#G$	$m1$	$m2$	$m3$	$m4$	$Isomorphism\ Type$
737280	0	0	4	5	$2^{10} : S_6$
31200	0	2	5	6	PGL(2,25)
80640	0	3	4	7	PGL(3,4):2
46080	0	4	4	5	$2^6 : S_6$

```
|N|=48
G<v,w,x,y,z,t>:=Group<v,w,x,y,z,t|v^2,w^3,x^2,y^2,
z^2, (w^-1*v)^2, w^-1*x*w*y, w*x*w^-1*z,v*x*v*y,
(x*y)^2,
t^2,
(t,v * x * y * z * w^-1),(t,x),
 (v*t)^m1,( x * y * z*t)^a,(x*t)^b,
(x*t^v)^c,(x*y*t^(x*w^2))^d,
(x*y*t)^e,(v*t^(x*w^2))^f,
(v*t^(w^2*y))^g,(v*t)^h,
(v*w*y*t^v)^i,( v * w * y*t)^j,
(v * w * y*t)^k,(w*t)^l,(w*t^y)^m,
(v*x*t^(x*w^2))^n,(v*x*t)^o,(v * w * y * z*t^v)^p,
(v * w * y * z*t)^q,(w*y*t)^r,(w*y*t^y)^s>;
```

Table 10.9: $2^{*12} : (S_4 \times 2)$

$\#G$	a	b	c	...	p	q	r	s	$m1$	$Isomorphism\ Type$
28800	0	0	0	...	0	0	0	4	5	$2 :^{\bullet} (A_5 : (S_5 \times 2))$
2880	0	0	0	...	0	0	0	4	4	$4 : S_6$
57600	0	0	0	0	0	0	4	0	4	$4 :^{\bullet} (A_5 : (S_5 \times 2))$
1440	0	0	0	0	0	0	4	5	6	$2^{\bullet} S_6$
983040	0	0	0	0	0	0	5	5	5	$2^{14} : S_6$
14400	0	0	0	0	0	0	6	5	4	$A_5 : (S_5 \times 2)$
368640	0	0	0	0	0	5	5	0	4	$2^9 : S_6$
23040	0	0	0	0	0	5	5	4	8	$2^5 : S_6$
30720	0	0	0	0	0	5	5	5	5	$2^9 :^{\bullet} A_5$
14400	0	0	0	0	0	6	0	4	5	$A_5 :^{\bullet} (S_5 \times 2)$
4769856	0	0	0	0	0	7	0	4	7	$L_2(13) :^{\bullet} (PGL_2(13) \times 2)$
240	0	0	0	0	2	0	0	5	10	$2^{\bullet} A_5$
672	0	0	0	0	2	0	0	7	8	$PGL_2(7)$
1008	0	0	0	0	2	0	0	7	9	$PSL_2(8)$
1344	0	0	0	0	2	0	0	8	8	$2^{\bullet} PGL_2(7)$
8640	0	0	0	0	2	0	0	8	10	$6 :^{\bullet} S_6$
6840	0	0	0	0	2	0	0	9	9	$PSL_2(19)$
41040	0	0	0	0	2	0	0	9	10	$S_3 : PSL_2(19)$
164160	0	0	0	0	2	0	0	10	9	$6 : PSL_2(19)$
69120	0	0	0	0	3	0	0	0	5	$2^5 : (3 : S_6)$
93600	0	0	0	0	3	0	0	6	6	$6 :^{\bullet} PGL_2(25)$
11520	0	0	0	0	3	5	5	0	8	$2^4 : S_6$
33592320	0	0	0	0	3	8	6	6	10	
61440	0	0	0	0	4	0	5	5	5	$2^{10} : A_5$
31457280	0	0	0	0	4	0	5	10	5	
805306368	0	0	0	0	4	4	0	0	8	
30720	0	0	0	0	4	5	5	5	5	$2^9 :^{\bullet} A_5$
1006632960	0	0	0	0	4	10	0	5	5	
322560	0	0	0	0	5	0	8	8	4	$2^2 :^{\bullet} (PGL_3(4) : 2)$
3686400	0	0	0	0	5	6	0	4	10	$2^8 :^{\bullet} (A_5 : (S_5 : 2))$
80640	0	0	0	0	5	7	8	8	4	$PGL_3(4) : 2$
1572864	0	0	0	0	8	6	0	4	6	$2^{15} :^{\bullet} S_4$
47185920	0	0	0	0	6	0	5	8	4	

10.8 Homomorphic Images of $2^{*30} : (S_5)$

```
|N|=120
G<x,y,t>:=Group<x,y,t|(x^-1 * y^-2)^2,(x * y^-2)^2,
y^-1*x^4*y*x^2,x^-1*y^-1*x^-1*y^-1*x^-1 *y*x*y,
y*x^-1*y^-1*x^3*y^4,
t^2,
(t,y * x^-1 * y),
(t,x^5),(t,x*y*x * y^-2),(x^5*t)^a,(x^5*t^y)^b,
(x^3 * y * x^-1*y*t^x)^c,( x^3 * y * x^-1 * y*t)^d,
(y*x^-1*y*t)^e,(y*x^-1*y*t^(y^-4))^f,(y*x^-1*y*t^x)^g,
(y * x^-1 * y*t^y)^h,(y^4*t)^i,(y^3*t^x)^j,(y^3*t)^k,
(y^-3*t^x)^l,(y^-3*t)^m,(x*y*t^(y^5))^n,(x*y*t^y)^o,
(x*y*t^(y^2))^p,(x*y*t)^q,( y^-1 * x^-1*t^(y^5))^r,
( y^-1 * x^-1*t^y)^s,( y^-1 * x^-1*t^(y^2))^u,
( y^-1 * x^-1*t)^v,(x^2*t)^w,(y^2*t^x)^z,
(y^2*t)^a1,(x*t)^b1,(x*t^y)^c1,(y*t^x)^d1,
(y*t)^e1,(x^3 * y^-1 * x*t^x)^f1,
(x^3 * y^-1 * x*t)^g1,(x * y * x * y^-1*t)^h1>;
```

Table 10.10: $2^{*30} : (S_5)$

$\#G$	a	b	c	...	$c1$	$d1$	$e1$	$f1$	$g1$	$h1$	$Isomorphism\ Type$
96	0	0	0	...	0	0	0	0	0	3	$2^2 : S_6$
960	0	0	0	...	0	0	0	0	4	4	$2^3 : S_6$
1200	0	0	0	...	0	0	0	0	4	10	$5^2 : 2^3$
86400	0	0	0	...	0	0	0	0	5	0	$2^{\bullet}((A_5 : A_6) : 2)$
120	0	0	0	...	0	0	0	0	5	5	$2^{\bullet}A_5$
1440	0	0	0	...	0	0	0	0	5	6	S_6
483840	0	0	0	...	0	0	0	0	6	4	$6 : (2(: S_8))$
125829120	0	0	0	...	0	0	0	4	8	4	
80640	0	0	0	...	4	0	0	0	7	0	$2 \times S_8$
25804800	0	0	0	...	4	0	0	4	0	0	$2^5 \times 5 : (4 : S_8)$
1612800	0	0	0	...	4	0	0	4	0	10	$2 \times 5 : (4 : S_8)$
60	0	0	0	...	5	0	0	10	5	5	A_5

10.9 Homomorphic Images of $Z_{10} \wr Z_3$

```
|N|=3000
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^10,y^10,z^10,w^3,
(x,y),(x,z),(y,z),x^-1*w^-1*z*w,y^-1*w^-1*x*w,
t^2,
```

```
(t,y),(t,z),(w*t)^a,(w^-1*t)^b,(x^2*t)^c,(x*t)^d,
(x^-1*t)^e,(x*y*t)^f,(x*w*t)^g>;
```

Table 10.11: $Z_{10} \wr Z_3$

$\#G$	a	b	c	d	e	f	g	$Isomorphism\ Type$
2400	0	0	0	0	0	0	3	$2^{\bullet}S_3$
48000	0	0	0	0	0	0	4	$2^4 : S_4$
96	0	0	0	0	0	2	3	$2^{\bullet}S_3$
48	0	0	0	0	0	2	4	S_4
120	0	0	0	0	0	2	5	A_5
648	0	0	0	0	0	3	6	$3^3 :^{\bullet} (S_4)$
384	0	0	0	0	0	4	4	$2^4 :^{\bullet} S_4$
3840	0	0	0	0	0	4	5	$2^5 : S_5$
75000	0	0	0	0	0	5	6	$5^3 : S_4$
8232	0	0	0	0	0	7	6	$7^3 :^{*} (S_4)$
17496	0	0	0	0	0	9	6	$3^6 :^{\bullet} (S_4)$
6000	0	0	0	0	0	10	4	S_4
1875000	0	0	0	0	0	10	5	$5^5 : A_5$
5040	0	0	0	0	3	0	7	S_7
117600	0	0	0	0	6	3	8	$PGL(2, 49)$
3000	0	0	0	2	0	5	6	$5^2 : S_5$

```
|N|=52
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,y^-1*x^-3*y*x^-2,
t^2,
(t,x^-1 * y^-1 * x), (y * x^2 * y*t)^r,
((x * y)^2*t^(x * y^-1))^a,
((x * y)^2*t)^b,((x * y)^2*t^(y^-1))^c,
 ((x * y)^2*t^(x^-1))^d,
(x * y*t^(x * y^-1))^e,(x * y*t)^f,(x * y*t^(y^-1))^g,
(x * y*t^(x^-1))^h,(y^-1 * x^-1*t^(x * y^-1))^i,
(y^-1 * x^-1*t)^j,
(y^-1 * x^-1*t^(y^-1))^k,
(y^-1 * x^-1*t^(x^-1))^l,
 (x*t)^m,(x^2*t)^n, (x^4*t)^o>;
```

Table 10.12: $Curtis\ Lemma\ 2^{*13} : (13 : 4)$

$\#G$	a	b	c	...	m	o	r	$Isomorphism\ Type$
18720000	0	0	0	...	5	0	4	$4^{\bullet}S_4(5)$

10.10 Homomorphic Images of $53^{*4} :_m (13 : 4)$

```
|N|=52
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2, x^(-13),
t^53,
t^(x^6)=t^28,(y*(t^5)^(x^2))^r, (y*(t^3)^(x * y))^s,
(y*(t^3)^(y * x))^u, (y*(t^12)^(x^-2))^v,(y*(t^5)^(x^-2))^w,
(y*(t^12)^(x*y))^z, (y*t^(x * y^-1))^aa,  (y*(t^4)^(x^2))^bb,
(y*(t^3)^(x^3))^cc, (y*(t^12)^(x^-1))^dd, (y*(t^10)^(x))^ee,
(y*(t^7)^(x^-1))^ff,(y*(t^2)^(x^-2))^gg, (y*(t^2)^(x^-1))^hh,
(y*(t^3)^(x^2 * y))^ii, (y*(t^8)^(x * y))^jj,(y*(t^34)^(y))^kk,
(y*(t^2)^(x^3))^ll, (y*(t^10)^(x^2))^mm,(y*(t^6)^(x^-2))^nn,
(y*(t^6)^(y^-1 * x))^oo,(y*t^(x^2 * y^-1))^pp,
(y*(t^6)^(x^-1))^qq, (y*(t^41)^(y^2))^rr>;
```

Table 10.13: $53^{*4} :_m (13 : 4)$

$\|G\|$	r	s	u	v	...	oo	pp	qq	rr	Isomorphism Type
74412	0	0	0	0	...	0	0	0	3	$PSL_2(53)$
1934712	0	0	0	0	...	0	0	0	4	$(13 : PGL_2(53))$

Appendix A

MAGMA Code for First Order Relations

```
NumberOfPrimitiveGroups(13);

N:=PrimitiveGroup(13,4);

#N;
N;
/*
Here we have the generators of the group
Permutation group N acting on a set of cardinality 13
Order = 52 = 2^2 * 13
    (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
    (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7)
*/

CompositionFactors(N);
/* G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(13)
    1
The isomorphism type of my control group is congruent to13:4
*/
```

```
%--------------------------------------------------------
Generators(N);
/*
    (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13),
    (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7)
*/

N.1;
N.2;
/* now we store each generator as x and y*/
S:=Sym(13);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=S!(1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7);
N:=sub<S|aa,bb>;

#N;
/*52*/

IsAbelian(N);
/*false*/
NL:=NormalLattice(G1);
NL;

/*Normal subgroup lattice
-----------------------

[4]  Order 52  Length 1  Maximal Subgroups: 3
---
[3]  Order 26  Length 1  Maximal Subgroups: 2
---
[2]  Order 13  Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
*/
Center(N);
/*Permutation group acting on a set of cardinality 13
Order = 1
the group doesn't have a center*/

FPGroup(N);

/*Finitely presented group on 2 generators
Relations
    $.2^4 = Id($)
```

```
    $.2^-2 * $.1^-1 * $.2^2 * $.1^-1 = Id($)
    $.2^-1 * $.1^-3 * $.2 * $.1^-2 = Id($)
*/
/*We write a presentation for N in terms of the generators*/
NN<x,y>:=Group<x,y|y^4,y^-2x^-1y^2x^-1,y^-1x^-3yx^-2>;
#NN;
/*52*/

/*the schreierSystem is used to convert permutations
 into words.*/
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..52]];
 for i in [2..52] do
 P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=aa; end if;
 if Eltseq(Sch[i])[j] eq -1 then P[j]:=aa^-1; end if;
 if Eltseq(Sch[i])[j] eq 2 then P[j]:=bb; end if;
 if Eltseq(Sch[i])[j] eq -2 then P[j]:=bb^-1; end if;
 end for;
 PP:=Id(N);
 for k in [1..#P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;

%-------------------------------------------
/*we stabilize and element from N, in this case
I stabilized 1, we could of stabilized a different number*/
N1:=Stabiliser(N,1);
#N1;
/*4*/
N1;

/*Permutation group N1 acting on a set of cardinality 13
Order = 4 = 2^2
    (2, 9, 13, 6)(3, 4, 12, 11)(5, 7, 10, 8)
*/

/* we used the following loop to convert the
permutation of the stabilizer of 1and we converted
them into words.*/

for i in [1..52] do if ArrayP[i] eq
```

```
N!(2, 9, 13, 6)(3, 4, 12, 11)(5, 7, 10, 8) then Sch[i];
end if; end for;

/*y^x, add it to the presentation of G and make sure it
commutes with t*/

G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,y^x)>;

#G;

%----------------------------------------------------
?*now we stabilize 1 and 2 and find the classes of N*/
N12:=Stabiliser(N,[1,2]);

Cent:=Centraliser(N,N12);

Cent;

/*Permutation group N acting on a set of cardinality 13
Order = 52 = 2^2 * 13
    (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
    (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7)
*/

C:=Classes(N);
#C;

for i in [1..#C] do

i, C[i][3];
end for;
/*1 Id(N)
2 (1, 6)(2, 5)(3, 4)(7, 13)(8, 12)(9, 11)
3 (1, 3, 6, 4)(2, 11, 5, 9)(7, 12, 13, 8)
4 (1, 4, 6, 3)(2, 9, 5, 11)(7, 8, 13, 12)
5 (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
6 (1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)
7 (1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)
*/


/*the following loop computes the orbits of the centralizer
of N and the classes.
```

```
for i in [2..7] do i, Orbits(Centraliser(N,C[i][3]));
end for;

/*2 [
    GSet{@ 10 @},
    GSet{@ 1, 6, 4, 3 @},
    GSet{@ 2, 5, 9, 11 @},
    GSet{@ 7, 13, 8, 12 @}
]
3 [
    GSet{@ 10 @},
    GSet{@ 1, 3, 6, 4 @},
    GSet{@ 2, 11, 5, 9 @},
    GSet{@ 7, 12, 13, 8 @}
]
4 [
    GSet{@ 10 @},
    GSet{@ 1, 4, 6, 3 @},
    GSet{@ 2, 9, 5, 11 @},
    GSet{@ 7, 8, 13, 12 @}
]
5 [
    GSet{@ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 @}
]
6 [
    GSet{@ 1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12 @}
]
7 [
    GSet{@ 1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10 @}
]
*/

/*and this loop converts the permutations(classes) into words
since our presentations has be written in terms of the
generators x and y*/
for j in [2..7] do for i in [1..52] do if ArrayP[i] eq
C[j][3] then C[j][3],Sch[i];

end if;

end for;

end for;
```

```
/*(1, 6)(2, 5)(3, 4)(7, 13)(8, 12)(9, 11)
(x * y)^2
(1, 3, 6, 4)(2, 11, 5, 9)(7, 12, 13, 8)
x * y
(1, 4, 6, 3)(2, 9, 5, 11)(7, 8, 13, 12)
y^-1 * x^-1
(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
x
(1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12)
x^2
(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)
x^4
*/
/*Now we factor the presentation of G by all
 first order relations and we run it on MAGMA */
for l,m,n,o,p,q,c,d,e,f,g,h,i,j,k in [0..10] do
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,y^x),
((x*y)^2*t^(x^-4))^c,
((x*y)^2*t)^d,
((x*y)^2*t^x)^e,
((x*y)^2*t^(x^6))^f,
((x*y)*t^(x^-4))^g,
((x*y)*t)^h,
((x*y)*t^x)^i,
((x*y)*t^(x^6))^j,
((y^-1*x^-1)*t^(x^-4))^k,
((y^-1*x^-1)*t)^l,
((y^-1*x^-1)*t^x)^m,
((y^-1*x^-1)*t^(x^6))^n,
(x*t)^o,
(x^2*t)^p,
(x^4*t)^q>;

/*this is used to find second order relations*/
Orbits(Stabiliser(Centraliser(N,xx*yy),1));
```

Appendix B

MAGMA Code for $S_z(8)$ DCE

```
/* This code guides a double coset enumeration
of G over NSz(8) DCE 2^{*}13:(13,4),
when n=5 and p=5 and factor by the center*/
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,y^x),
 ((y^-1*x^-1)*t^(x^6))^5,
 (x^2*t)^5>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
 c:=Center(G1).1;
Order(c);
 NN:=G;
 N:=G1;
 Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..58240]];
 for i in [2..58240] do
 P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
 if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
 if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
 if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
 end for;
 PP:=Id(N);
for k in [1..#P] do
```

```
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;
 for i in [1..58240] do if ArrayP[i] eq c then Sch[i];
 end if; end for;


%--------------------------------------------------

/*Add relation to the  progenitor so it is being
factored by the center*/
G<x,y,t>:=Group<x,y,t|y^4,y^-2*x^-1*y^2*x^-1,
y^-1*x^-3*y*x^-2,t^2,(t,y^x),
((y^-1*x^-1)*t^(x^6))^5,
(x^2*t)^5,x^3*t*x^-2*t*y^-1*t*y*t*y^2*t>;
#G;

S:=Sym(13);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=S!(1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7);
N:=sub<S|xx,yy>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);

\*now label your t's in terms of x and y*/

ts:=[Id(G1): i in [1..13]];
ts[1]:=f(t); ts[2]:=f(t^x);
ts[3]:=f(t^(x^2));
ts[4]:=f(t^(x^3));
ts[5]:=f(t^(x^4));
ts[6]:=f(t^(x^5));
ts[7]:=f(t^(x^6));
ts[8]:=f(t^(x^7));
ts[9]:=f(t^(x^8));
ts[10]:=f(t^(x^9));
ts[11]:=f(t^(x^(10)));
ts[12]:=f(t^(x^(11)));
ts[13]:=f(t^(x^(12)));


DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
```

```
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations
Q[I] applied sequentially.
*/
 v := pt;
 for i in I do
   v := v^(Q[i]);
 end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
  for i := 1 to 13 do
   cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..560] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

%--------------------------------------------------

N1:=Stabiliser(N,[1]);
 SSS:={[1]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[1] eq n*ts[Rep(Seqq[i])[1]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N1; #N1;
T1:=Transversal(N,N1);
 for i in [1..#T1] do ss:=[1]^T1[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..560] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N1);
```

```
%--------------------------------------------------

N12:=Stabiliser(N,[1,2]);
 SSS:={[1,2]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N12; #N12;
T12:=Transversal(N,N12);
 for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..560] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N12);
%--------------------------------------------------

N13:=Stabiliser(N,[1,3]);
 SSS:={[1,3]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N13; #N13;
T13:=Transversal(N,N13);
 for i in [1..#T13] do ss:=[1,3]^T13[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..560] do if cst[i] ne [] then m:=m+1;
```

```
end if; end for; m;
Orbits(N13);
%--------------------------------------------------

N15:=Stabiliser(N,[1,5]);
 SSS:={[1,5]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[1]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N15; #N15;
T15:=Transversal(N,N15);
 for i in [1..#T15] do ss:=[1,5]^T13[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..560] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N15);
%--------------------------------------------------

/* Continue inserting the code from below
,if m increases by a value, then we have a new
double coset. For single cosets the value of m tells us
the number of single cosets for every new double coset.

N124:=Stabiliser(N,[1,2,4]);
 SSS:={[1,2,4]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[1]*ts[2]*ts[4] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
```

```
 end if;
 end for;
 end for;

N124s:=N124;
for n in N do if 1^n eq 2 and 2^n eq 1 and 4^n eq 12 then
N124s:=sub<N|N124s,n>;
 end if; end for;
 #N124s;
N124s;
[1,2,4]^N124s;

N124:=Stabiliser(N,[1,2,4]);
N124;
 N124:=sub<N|(1, 2)(3, 13)(4, 12)(5, 11)(6, 10)(7, 9)>;
#N124;
[1,2,4]^N124;

T:=Transversal(N,N124);
for i in [1..#T] do {[1,2,4]^N124}^T[i];
 end for;
for n in IN do if ts[2]*ts[1]*ts[12] eq
n*ts[1]*ts[2]*ts[4] then n; end if; end for;
for m,n in IN do if ts[2]*ts[1]*ts[12] eq
m*(ts[1]*ts[2]*ts[4])^n then A:=m; B:=n;
end if; end for;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
g:=function(W);
g(G);
ts[2]*ts[1]*ts[12] eq f(x^2)*ts[1]*ts[2]*ts[4];

T124:=Transversal(N,N124);
for i in [1..#T124] do ss:=[1,2,4]^T124[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..560] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
Orbits(N124);

/*since there are equal coset names on the double coset
 we check all elements in N that sends [2112] to [124] we
 do the loop listen below to determine the relation.*/
```

```
 /*we continue using the loops until the number of m
 increases to 560, since the  index of |G|/|N|=560*/


%--------------------------------------------------


/* Since we have all new existing double cosets
now we check for those double cosets where m did
not increased. We verify if they are equal to other existing
double cosets. We use the following code*/

for m,n in IN do if ts[1]*ts[2]*ts[8]*ts[1] eq
m*(ts[1]*ts[2]*ts[3])^n then m,n;  end if; end for;

/*since [1281] is not a new double coset we check if it is
equal to other existing double cosets. In this case is
equal to [123]. We use the loop listen below to test
every new double coset until MAGMA prints out permutations
so we know the coset [1281] lives in the double coset [123].
Lastly, we find by what relation [1281] lives in [123]*/

for m,n in IN do if ts[1]*ts[2]*ts[8]*ts[1] eq
m*(ts[1]*ts[2]*ts[3])^n then A:=m;B:=n;  end if; end for;
W,phi:=WordGroup(G1);
 rho:=InverseWordMap(G1);
 A@rho;
g:=function(W);
w4 := W.1 * W.2; return w4;
end function;
g(G);
ts[1]*ts[2]*ts[8]*ts[1] eq f(x*y)*ts[10]*ts[11]*ts[12];



%--------------------------------------------------

/*Once we have found all orbits and the order of m has
increased to 560 our group is closed under right
 multiplication and a Cayley graph is given to
 summarize the work.*/
```

Appendix C

MAGMA Code for $2 \times PGL_2(27)$ over M=(13:2) DCE

```
/* To complete a DCE of G over a maximal subgroup M we
follow a similar procedure for a regular DCE of G. But in
this case for one of the loops we must replace M with N.

%-------------------------------------------------------

S:=Sym(13);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=S!(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7);
N:=sub<S|xx,yy>;

G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-13,t^2,(t,y * x^2),
((x^4)*t*t^x)^3,
 ((x^6)*t*t^x)^2>;

H:=sub<G|x,y,y * t * x^4 * t * x * t * y * t,
t * x^4 * y * t * x^4 * y * t * x^3 * y>;
 #H;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
IM:=sub<G1|f(x),f(y),f(y * t * x^4 * t *x* t * y * t),
f(t * x^4 * y * t * x^4 * y* t * x^3 * y)>;
#IN;#IM;

ts:=[Id(G1): i in [1..13]];
```

```
ts[1]:=f(t); ts[2]:=f(t^x);
ts[3]:=f(t^(x^2));
ts[4]:=f(t^(x^3));
ts[5]:=f(t^(x^4));
ts[6]:=f(t^(x^5));
ts[7]:=f(t^(x^6));
ts[8]:=f(t^(x^7));
ts[9]:=f(t^(x^8));
ts[10]:=f(t^(x^9));
ts[11]:=f(t^(x^(10)));
ts[12]:=f(t^(x^(11)));
ts[13]:=f(t^(x^(12)));

DoubleCosets(G,H,sub<G|x,y>);
Index(G,sub<G|x,y>);


prodim := function(pt, Q, I)

 v := pt;
 for i in I do
   v := v^(Q[i]);
 end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
  for i := 1 to 13 do
   cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..1512] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

%-------------------------
N1:=Stabiliser (N,[1]);
SSS:={[1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do for n in IM do if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
```

```
end if; end for; end for;
N1; #N1;
T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..378] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1);

%------------------------
N12:=Stabiliser(N,[1,2]);
 SSS:={[1,2]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N12; #N12;

/* Equal Coset Name [1,2]~[9,8]
 in one of the following loops we have n in M not N*/

N12s:=N12;
for n in M do if 1^n eq 9 and 2^n eq 8  then
N12s:=sub<N|N12s,n>;
 end if; end for;
 #N12s;
N12s;
[1,2]^N12s;

N12:=Stabiliser(N,[1,2]);
N12;
 N12:=sub<N|(1, 9)(2, 8)(3, 7)(4, 6)(10, 13)(11, 12)>;
#N12;
[1,2]^N12;
```

```
T:=Transversal(N,N12);
for i in [1..#T] do {[1,2]^N12}^T[i];
 end for;
for n in IM do if ts[1]*ts[2] eq
n*ts[9]*ts[8] then n; end if; end for;

ts[1]*ts[2] eq f(x)*ts[9]*ts[8];

/* Now we find the relation that sends [1,2] to [9,8]*/

for n in IM do if ts[1]*ts[2]
eq n*ts[9]*ts[8] then n; end if; end for;
for n in IM do if ts[1]*ts[2] eq
n*ts[9]*ts[8] then A:=n; end if; end for;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
g:=function(W);
    return W.1;
end function;
g(G);
T12:=Transversal(N,N12);
 for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1512] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N12);

%---------------------------------------------------

N13:=Stabiliser(N,[1,3]);
 SSS:={[1,3]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
```

```
N13; #N13;
T13:=Transversal(N,N13);
 for i in [1..#T13] do ss:=[1,3]^T13[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1512] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
Orbits(N13);

%-------------------------------------------------

N14:=Stabiliser(N,[1,4]);
 SSS:={[1,4]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[4] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N14; #N14;
T14:=Transversal(N,N14);
 for i in [1..#T14] do ss:=[1,4]^T14[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1512] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
Orbits(N14);

%-------------------------------------------------

N16:=Stabiliser(N,[1,6]);
 SSS:={[1,6]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[6] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
```

```
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N16; #N16;
T16:=Transversal(N,N16);
 for i in [1..#T16] do ss:=[1,6]^T16[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1512] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N16);


/*Every time the value of m increases we have a new
double coset.  But keep in mind that he value of m is
the number of single cosets found on each double coset.
We continue until the value of m increases to the number
of the index, 1512. Once we have found all 22 DC now we
check where the not existing double cosets live.*/


%---------------------------------------------------
/*If the value of m does not increase
then is not a new double coset so
now we check for the the non existing double
coset using the following loops*/

DoubleCosets(G,H,sub<G|x,y>);

/*we use the loop above to label all 22 double
cosets including identity(loop below). */

A:=[Id(G1): i in [1..21]];
 A:=[Id(G1): i in [1..21]];
 A[1]:=f(t * y * t * x^2 * t * y * t);
 A[2]:=f(t * x^2 * y * t * x * y * t);
 A[3]:=f(t * x^3 * t * x * y *t);
 A[4]:=f(t * x * t * x^-1 * t * y * t);
 A[5]:=f(t * x * t* x * t * x * t);
 A[6]:=f(t * x^2 * y * t * y * t);
 A[7]:=f(t * x^3 * t * y * t);
 A[8]:=f(t * y * t * x * y * t);
 A[9]:=f(t * y * t * x * t);
```

```
 A[10]:=f(t * x * t * x^-1 * t);
 A[11]:=f(t * x * t * y * t);
 A[12]:=f(t); A[13]:=f(t * x * t);
 A[14]:=f(t * y * t);
 A[15]:=f(t * x * y * t);
 A[16]:=f(t * x * t * x * t);
 A[17]:=f(t * y * t * y * t);
 A[18]:=f(t* x^2 * t * y * t);
 A[19]:=f(t * x * y * t * y * t);
 A[20]:=f(t * x * y * t * x^-1 * t);
 A[21]:=f(t * x * t * x * y * t);

%--------------------------------------------------
/* once we have label all 22 double cosets we
continue and use the following loop. This loop
lets us know where the exactly the double cosets
live. To verify if [1,2] is new we use the loop shown
 below*/

 for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[2] eq m*(A[i])^n then "true"; end if;
end for; end for; end for;
 for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[2] eq m*(A[i])^n then i; end if; end for;
end for; end for;
/*13*/

/*the loop tells us that is on 13, note 13 is a
 double coset of two t's*/

for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[3] eq m*(A[i])^n then "true"; end if;
end for; end for; end for;
for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[3] eq m*(A[i])^n then i; end if; end for;
 end for; end for;
/*14*/

 for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[4] eq m*(A[i])^n then "true"; end if;
end for; end for; end for;
for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[4] eq m*(A[i])^n then i; end if; end for;
 end for; end for;
```

```
/*15*/

 for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[5] eq m*(A[i])^n then "true"; end if;
end for; end for; end for;
for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[5] eq m*(A[i])^n then i; end if; end for;
end for; end for;
/*14*/

/*Note,  [1,5] is not a new double coset since
in 14 we have the double coset [1,3] so [1,5]
 lives in [1,3] . Now we find by what relation
 they are equal. We are going to use the
 SchreierSystem*/

%---------------------------------------------------
/* Since we have G over M we have aded on
the loop g in Mand h in N*/

for g in IM do for h in IN do if
ts[1]*ts[5] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if
ts[1]*ts[5] eq g*(ts[1]*ts[3])^h
 then A:=g;B:=h; break; end if;
 end for; end for;


N:=G1; NN:=G;
 Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
 P:=[Id(G1): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x^-1); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
end for;
 PP:=Id(G1);
 for k in [1..#P] do PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
end for;
```

```
 for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
 end if; end for;
/*x^4 * y * t * x^4 * t * x * t * y * t*/
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
/*conjugate [1,3] by x^4*y*/
ts[1]*ts[5] eq
f(x^4 * y * t * x^4 * t * x * t * y * t)*ts[8]*ts[6];

 %----------------------------------------------------
/* lets try one more using the label of each
double coset and SchreierSystem */

for i in [1..21] do for m in IM do for n in IN do
if ts[1]*ts[3]*ts[9] eq m*(A[i])^n then i; end if;
end for; end for; end for;
/*Note 10 is my new double coset name [1,6]*/

for g in IM do for h in IN do if ts[1]*ts[3]*ts[9] eq
g*(ts[1]*ts[6])^h then g,h;
break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]*ts[9] eq
 g*(ts[1]*ts[6])^h then A:=g;B:=h; break; end if;
 end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
/*conjugate [1,6] by x^-3*/
ts[1]*ts[3]*ts[9] eq
f(t * x^4 * y * t * x^4 * y * t * x^3)*ts[11]*ts[3];


%---------------------------------------------------
/*Once we have found all orbits and the order of m has
increased to 1512 our group is closed under right
 multiplication and a Cayley graph is given to
 summarize the work.
```

Appendix D

MAGMA Code for Monomial Progenitor $53^{*2} :_m (13 : 4)$

```
G:=TransitiveGroup(13,2);
#G;
G;
G.1;
G.2;
G eq sub<G|G.1,G.2>;

%-----------------------------------------------------

xx:=G!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13);
yy:=G!(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7);
G eq sub<G|xx,yy>;
IsAbelian(G);
/*False, Continue with the process*/
CG:=CharacterTable(G);
CG;

%-----------------------------------------------------
/*Now we find the classes of G*/

C:=Classes(G);
C;

Class(G,C[1][3]);
/* find all 8 classes by changing C[i][3] for i in 1,...,8*/
```

```
CG[3];

%---------------------------------------------------
/*Now find the subgroup that is equal to the index,2*/

S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 2 then i;
end  if; end for;
H:=S[3]`subgroup;
CH:=CharacterTable(H);
CH;


for i in [2..13] do if Induction(CH[i],G) eq CG[3] then i;
end if; end for;


Induction(CH[4],G) eq CG[3];
I:=Induction(CH[4],G);
I eq CG[3];

%---------------------------------------------------
/* we use the following code to find all classes of H*/

Cprime:=Classes(H);
Class(H,Cprime[1][3]);

/*change Cprime[i][3]; for i in 1,...,8*/

CH[4];

%---------------------------------------------------
/**Find the 2 x 2 Matrix for xx and yy**/
C:=CyclotomicField(13);
GG:=GL(2,C);

T:=Transversal(G,H);
#T;
/* in here I have the index and the prime field*/
N:=H;

A:=[[C.1,0]:i in [1..2]];
 for i,j in [1..2] do A[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
```

```
A[i,j]:=CH[4] (T[i]*xx*T[j]^-1); end if; end for;
GG!A;
Order(GG!A);

B:=[[C.1,0]:i in [1..2]];
for i,j in [1..2] do B[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[4] (T[i]*yy*T[j]^-1); end if; end for;
GG!B;
Order(GG!B);

%--------------------------------------------------

C:=CyclotomicField(13);
GG:=GL(2,53);
/* in here I have the index and the prime field*/
N:=H;
T:=Transversal(G,H);
#T;
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^5 then D[i,j]:=24;
end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^8 then D[i,j]:=42;
end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
A:=[[0,0]: i in [1..2]];
mat(4,xx,A,2);
mat(4,yy,A,2);

AA:=GG!mat(4,xx,A,2);
BB:=GG!mat(4,yy,A,2);
HH:=sub<GG|AA,BB>;
IsIsomorphic(HH,G);


%--------------------------------------------------

/***Find the permutation representation with #Field -1*/

C:=CyclotomicField(52);
```

```
GG:=GL(2,C);
/* in here I have the index and the prime field*/
N:=H;
A:=[[C.1,0]:i in [1..2]];  A:=[[C.1,0]: i in [1..2]];
 for i,j in [1..2] do A[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[4] (T[i]*xx*T[j]^-1); end if; end for;
GG!A;
Order(GG!A);

B:=[[C.1,0]:i in [1..2]];  B:=[[C.1,0]: i in [1..2]];
 for i,j in [1..2] do B[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[4] (T[i]*yy*T[j]^-1); end if; end for;
GG!B;
Order(GG!B);

HH:=sub<GG|A,B>;
IsIsomorphic(HH,G);

%--------------------------------------------------

perm := function(n, p, mat)
/* Return the matrix converted to permutation
of S_{n*p}.*/
C<u>:=CyclotomicField(p);
Z:=Integers ();
 s:=[];
for i in [1..n] do
 s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do
j:=0; done:=0;
repeat
if z[1,i]/u^j in Z then
 if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
 done:=1;
end if; end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
```

```
for i in [1..(p-1)] do
for a in [1..n] do
 w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for; end for;
 S:=Sym(n*p);
 w:=S!w;
return w;
end function;
HH:=sub<Sym(2*52)|perm(2,52,GG!A),
perm(2,52,GG!B)>;
IsIsomorphic(HH,G);
perm(2,52,GG!A);

perm(2,52,GG!B);

%--------------------------------------------------

FPGroup(G);
G<x,y>:=Group<x,y|y^2,(x^-1*y)^2, x^(-13)>;
S:=Sym(104);
xx:=S!(1,47,91,87,97,19,55,71,31,25,93,29,83)
(3,95,77,69,89,39,5,37,63,51,81,59,61)
(7,85,49,33,73,79,11,75,21,103,57,13,17)
(9,27,35,15,65,99,67,41,53,23,45,43,101)
(2,84,30,94,26,32,72,56,20,98,88,92,48)
(4,62,60,82,52,64,38,6,40,90,70,78,96)
(8,18,14,58,104,22,76,12,80,74,34,50,86)
(10,102,44,46,24,54,42,68,100,66,16,36,28);

yy:=S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)
(15, 16)(17, 18)(19, 20)(21,22)(23, 24)(25, 26)
(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39,
40)(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)(51, 52)
(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)
(65, 66)(67, 68)(69, 70)(71, 72)(73, 74)(75,
76)(77, 78)(79, 80)(81, 82)(83, 84)(85, 86)(87, 88)
(89, 90)(91, 92)(93,
94)(95, 96)(97, 98)(99, 100)(101, 102)(103, 104);

N:=sub<S|xx,yy>;
#N;

Sch:=SchreierSystem(G,sub<G|Id(G)>);
 ArrayP:=[Id(N): i in [1..#N]];
```

```
 for i in [2..#N] do
 P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
 if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
 if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
 end for;
 PP:=Id(N);
 for k in [1..#P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;
Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,17,
19,21,23,25,27,29,31,33,35,37,39,41,43,45,47,49,
51,53,55,57,59,61,63,65,67,69,71,73,75,77,79,81,83,
85,87,89,91,93,95,97,99,101,103});

Normaliser;
Normaliser.1;

A1:=N!(1,47,91,87,97,19,55,71,31,25,93,29,83)
(2,84,30,94,26,32,72,56,20,98,88,92,48)(3,95,
77,69,89,39,5,37,63,51,81,59,61)(4,62,60,82,52,
64,38,6,40,90,70,78,96)(7,85,49,33,73,79,11,75,
21,103,57,13,17)(8,18,14,58,104,22,76,12,80,74,
34,50,86)(9,27,35,15,65,99,67,41,53,23,45,43,
101)(10,102,44,46,24,54,42,68,100,66,16,36,28);
for i in [1..26] do if ArrayP[i] eq A1 then Sch[i];
end if; end for;

G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2, x^(-13),t^53,
t^(x^5)=t^3,(t,t^y)>;

%------------- Find First Order Relation----/
C:=Classes(N);
#C;
for i in [2..8] do i, Orbits(Centraliser(N,C[i][3]));
end for;

for j in [2..8] do for i in [1..26] do if ArrayP[i] eq C[j][3]
then C[j][3],Sch[i];
end if;
end for;
end for;
```

Appendix E

MAGMA Code for Wreath Product

```
%*--------Producing a Wreath Product Progenitor-------------/
N:=TransitiveGroup(30,437);
PP<x,y,z,w>:=Group<x,y,z,w|x^10,y^10,z^10,w^3,(x,y),(x,z),
x^w=y,y^w=z,z^w=x>;
f3,P1,k3:=CosetAction(PP,sub<PP|Id(PP)>);
IsIsomorphic(G,P1);
/*true Mapping from: GrpPerm: G to GrpPerm: P1
Composition of Mapping from: GrpPerm: G to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: P1
*/
W:=WreathProduct(CyclicGroup(10),CyclicGroup(3));
PP<x,y,z,w>:=Group<x,y,z,w|x^10,y^10,z^10,w^3,(x,y),(x,z),
x^w=y,y^w=z,z^w=x>;
f3,P1,k3:=CosetAction(PP,sub<PP|Id(PP)>);

%-----------------------------------------------
S:=Sym(30);
xx:=S!(1,2,3,4,5,6,7,8,9,10);
yy:=S!(11,12,13,14,15,16,17,18,19,20);
zz:=S!(21,22,23,24,25,26,27,28,29,30);
ww:=S!(1,11,21)(2,12,22)(3,13,23)(4,14,24)(5,15,25)
(6,16,26)(7,17,27)(8,18,28)(9,19,29)(10,20,30);
N:=sub<S|xx,yy,zz,ww>;
N eq sub<N|xx,yy,zz,ww>;
CompositionFactors(N);
```

```
FPGroup(N);
/*1st Presentation*/
NN<x,y,z,w>:=Group<x,y,z,w|x^10,y^10,z^10,w^3,
(x,y),(x,z),(y,z),x^-1*w^-1*z*w,y^-1*w^-1*x*w>;
#NN;
/*3000*/

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
  ArrayP:=[Id(N): i in [1..3000]];
  for i in [2..3000] do
 P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
  if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
  if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
  if Eltseq(Sch[i])[j] eq -4 then P[j]:=ww^-1; end if;
 end for;
  PP:=Id(N);
  for k in [1..#P] do
  PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
  end for;

N1:=Stabiliser(N,1);

#N1;
/*100*/
N1;
/*(11, 12, 13, 14, 15, 16, 17, 18, 19, 20)
    (21, 22, 23, 24, 25, 26, 27, 28, 29, 30)*/

for i in [1..3000] do if ArrayP[i] eq N!(11, 12, 13,
 14, 15, 16, 17, 18, 19, 20)
 then Sch[i]; end if; end for;
/*y*/

for i in [1..3000] do if ArrayP[i] eq N! (21, 22, 23,
 24, 25, 26, 27, 28, 29, 30)
 then Sch[i]; end if; end for;
```

```
/*z*/

G<x,y,z,w,t>:=Group<x,y,z,w,t|x^10,y^10,z^10,w^3,
(x,y),(x,z),(y,z),x^-1*w^-1*z*w,y^-1*w^-1*x*w,t^2,(t,y),
(t,z)>;

%-------------------------------------------------
C:=Classes(N);

#C;
/*360*/
for i in [1..#C] do
i, C[i][3];
end for;

for i in [2..360] do i, Orbits(Centraliser(N,C[i][3]));
end for;

for j in [2..360] do for i in [1..13] do if ArrayP[i] eq
C[j][3] then C[j][3],Sch[i];
end if;
end for;
end for;
```

Appendix F

Finding Generators of $PSL(2,7)$

```
/**********Find Mapping for PSL(2,7)*********/
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^3,z^2,w^2,
(y^-1*x)^2, y^-1*z*y*w, (x*z)^2,(z*w)^2, y*z*y^-1*z*w,
t^2,(t,x*y),(x * y^-1 * w*t)^3,(y*t)^3,(w * t)^3,
 x * w * y * t * z * w * t,t * z * t * x * w * y * t * w * t>;
#G;


 S:=Sym(12);
 xx:=S!(1, 4)(2, 5)(3, 6)(8, 9)(10, 12);
 yy:=S!(1, 7, 4)(2, 8, 6)(3, 9, 5)(10, 12, 11);
 zz:=S!(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12);
 ww:=S!(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12);

 N:=sub<S|xx,yy,zz,ww>;
 #N;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
IN:=sub<G1|f(x),f(y),f(z),f(w)>;
 CompositionFactors(G1);

%-------------------------------------------------
/*we use the following loop to compute the
 single cosets of the group*/
for i in [1..7] do i, cst[i]; end for;
/*1 []
2 [ 12 ]
3 [ 10 ]
4 [ 11 ]
5 [ 9 ]
```

```
6 [ 8 ]
7 [ 5 ]
*/

%------------------------------------------------

/*Here we have the image of x,y,z,w, and t*/
f(x);
/*(2, 3)(5, 6)*/
f(y);
/*(2, 4, 3)(5, 7, 6)*/
f(z);
/*(2, 5)(3, 6)*/
f(w);
/*(2, 5)(4, 7)*/
f(t);
/*(1, 2)(3, 4)*/

%------------------------------------------------

/*this code prints out the twelve t's we have*/
f(t)^IN;
/*GSet{@
    (1, 2)(3, 4),
    (1, 3)(2, 4),
    (1, 4)(2, 3),
    (1, 5)(4, 6),
    (1, 5)(3, 7),
    (1, 6)(4, 5),
    (1, 3)(5, 7),
    (1, 4)(5, 6),
    (1, 7)(3, 5),
    (1, 2)(6, 7),
    (1, 6)(2, 7),
    (1, 7)(2, 6)
@}*/

/*rename f(t),f(x),f(y),f(z),f(w)*/
ft:=f(t);
fx:=f(x);
fy:=f(y);
fz:=f(z);
fw:=f(w);
```

```
%------------------------------------------------

/* we must find all twelve t's that satisfies
 the twelve permutationslisted above*/
ft:=f(t);
 f2:=(ft)^((fz*fy));
f3:=ft^((fx^fw));
f4:=(ft^(fx));
f5:=ft^(fw*fy);
f6:=ft^(fz);
f7:=ft^(fy);
f8:=ft^(fw*fx);
f9:=ft^(fw);
f10:=ft^(fx*fw^(-1));
f11:=ft^((fz*fx*fy^(-1))^3);
f12:=ft^(fz*fw);

/*check if we get all the 12 t's match
to the ones in f(t)^IN;*/
ft;
f2;
f3;
f4;
f5;
f6;
f7;
f8;
f9;
f10;
f11;
f12;

%------------------------------------------------

/***Finding Generators of PSL(2,7) ****/

G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^3,z^2,w^2,
(y^-1*x)^2, y^-1*z*y*w, (x*z)^2,(z*w)^2,
y*z*y^-1*z*w,t^2,(t,x*y),(x * y^-1 * w*t)^3,
(y*t)^3,(w * t)^3, x * w * y * t * z * w * t,
t * z * t * x * w * y * t * w * t>;
#G;
```

```
 S:=Sym(12);
 xx:=S!(1, 4)(2, 5)(3, 6)(8, 9)(10, 12);
 yy:=S!(1, 7, 4)(2, 8, 6)(3, 9, 5)(10, 12, 11);
 zz:=S!(1, 6)(2, 5)(3, 4)(7, 11)(8, 10)(9, 12);
 ww:=S!(1, 9)(2, 7)(3, 8)(4, 10)(5, 11)(6, 12);

 N:=sub<S|xx,yy,zz,ww>;
 #N;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
IN:=sub<G1|f(x),f(y),f(z),f(w)>;
 CompositionFactors(G1);

 %-----------------------------------------------

/* we are now in symmetric 8, menacing we are working
with 8 letters.
Also store alpha, beta, and gamma after finding by hand*/
S:=Sym(8);
alpha:=S!(7,1,2,3,4,5,6);
beta:=S!(1,4,2)(3,5,6);
gamma:=S!(7,8)(1,6)(2,3)(4,5);

psl:=sub<S|alpha, beta, gamma>;
IsIsomorphic(G1,psl);
/*true Homomorphism of GrpPerm: G1, Degree 7,
Order 2^3 * 3 * 7 into GrpPerm: psl,
Degree 8, Order 2^3 * 3 * 7 induced by
    (2, 3)(5, 6) |--> (1, 7)(2, 4)(3, 8)(5, 6)
    (2, 4, 3)(5, 7, 6) |--> (1, 6, 2)(4, 5, 7)
    (2, 5)(3, 6) |--> (1, 8)(2, 6)(3, 7)(4, 5)
    (2, 5)(4, 7) |--> (1, 2)(3, 4)(5, 7)(6, 8)
    (1, 2)(3, 4) |--> (1, 3)(2, 6)(4, 8)(5, 7)
on the left hand side we have the image of x,y,z,w,t.
and on the right hand side we have the homomorphism.
We are going to use the homomorphism to find the map
 for each one, this is done by hand.*/

 %-----------------------------------------------
f,G2,k:=CosetAction(S,sub<S|alpha,beta,gamma>);
IN:=sub<G2|f(alpha),f(beta),f(gamma)>;

/* now we store the homomorphisms as follows:*/
X:= S!(1, 7)(2, 4)(3, 8)(5, 6);
Y:= S!(1, 6, 2)(4, 5, 7);
```

```
Z:=S!(1, 8)(2, 6)(3, 7)(4, 5);
W:=S!(1, 2)(3, 4)(5, 7)(6, 8);
T:=S!(1, 3)(2, 6)(4, 8)(5, 7);

T; /*(1, 3)(2, 6)(4, 8)(5, 7)*/
T4:=T^(X);     /*=t_4=(1, 6)(2, 3)(4, 5)(7, 8),*/
T2:=T^(Z*Y); /*(1, 2)(3, 5)(4, 8)(6, 7)*/
T5:=T2^X; /*(1, 5)(2, 3)(4, 7)(6, 8)*/
T3:=T^(X^W); /*(1, 3)(2, 8)(4, 5)(6, 7)*/
T6:=T3^X; /*(1, 5)(2, 6)(3, 4)(7, 8)*/
T7:=T^Y; /*(1, 2)(3, 6)(4, 7)(5, 8)*/
T8:=T^(W*X); /*(1, 6)(2, 4)(3, 7)(5, 8)*/
T9:=T8^X; /*(1, 8)(2, 4)(3, 6)(5, 7)*/
T10:=T^(X*W^-1); /*(1, 4)(2, 8)(3, 7)(5, 6)*/
T11:=T^((Z*X*Y^-1)^3); /*(1, 4)(2, 7)(3, 5)(6, 8)*/
T12:=T10^X; /*(1, 8)(2, 7)(3, 4)(5, 6)*/
```

Appendix G

Maximal Subgroup of $PSL(2, 8)$ over $S_4 \times 2$

```
/*////////////////////Change Ker=2 to ker=1//////////////////*/
S:=Sym(12);
vv:=S!(1, 2)(3, 6)(4, 8)(5, 9)(7, 11);
ww:=S!(1, 3, 11)(2, 7, 6)(4, 8, 12)(5, 9, 10);
xx:=S!(2, 8)(3, 9)(7, 10)(11, 12);
yy:=S!(1, 4)(5, 6)(7, 12)(10, 11);
zz:=S!(1, 5)(2, 9)(3, 8)(4, 6);
N:=sub<S|vv,ww,xx,yy,zz>;
#N;
/*48*/
G<v,w,x,y,z,t>:=Group<v,w,x,y,z,t|v^2,w^3,x^2,y^2,z^2,
(w^-1*v)^2, w^-1*x*w*y, w*x*w^-1*z,v*x*v*y,(x*y)^2,
t^2,(t,v * x * y * z * w^-1),(t,x),(v*t)^7,
(v * w * y * z*t^v)^2,
(v * w * y * z*t)^0,(w*y*t)^0,(w*y*t^y)^9>;
#G;
/*1008*/
f,G1,k:=CosetAction(G,sub<G|v,w,x,y,z>);
IN:=sub<G1|f(v),f(w),f(x),f(y),f(z)>;
CompositionFactors(G1);

 %-------------------------------------------------
#k;
/*the kernel is 2 to have a better image we want to change
kernel to be 1 since we want only identity to live in K
otherwise we would have one thing be mapping to two
```

```
things which it wont be a one to this is the process to change
 the kernel to 1*/

  %------------------------------------------------
/*first we want to find the order the mapping of
each generator*/
f(x);
/*Id(G)*/
Order(f(y));
/*1*/
Order(f(z));
/*1*/
Order(f(v));
/*2*/
Order(f(w));
/*3*/

/*Noticed that the the order of f(x)=f(y)=f(z)=1.
Thus we are going to construct a new progenitor using the
 original progenitor but changing x,y, and z with identity.
  so we use the old progenitor to get a new one as follow*/
/* since x,y,& z are identity then we only use v=2 and w=3*/

 %------------------------------------------------

G<v,w,t>:=Group<v,w,t|v^2,w^3, (w^-1*v)^2,t^2,(t,v*w^-1),
(v*t)^7,(v * w * t^v)^2,(w*t)^9>;
#G;
/*the new order of G is half of what was our old order of G,
1008since we have a new G we do coset action of the
new group call it G1*/
f,G1,k:=CosetAction(G,sub<G|v,w>);
#k;
/*yes, the order of k is 1 now we can construct a better
 image for this group*/

 %------------------------------------------------
CompositionFactors(G1);
/*we stil have  G
    |  A(1, 8)                    = L(2, 8)
    1
*/
#sub<G|v,w>;
/*6*/
```

```
NN<v,w>:=Group<v,w|v^2,w^3, (w^-1*v)^2>;
HH:=sub<NN|v*w^-1>;
/*2*/

 %------------------------------------------------
/*now we do the point stabilizer of 1*/
ff,N,k1:=CosetAction(NN,HH);
 N;
/*Now we get the generators of S_3
Symmetric group N acting on a set of cardinality 3
Order = 6 = 2 * 3
 (1, 2)
 (1, 2, 3)*/
 ff(v*w^-1);
/*(2, 3)*/
Stabiliser(N,1);

 %------------------------------------------------

/*/////Now we start with the process of maximal subgroup///*/
S:=Sym(12);
xx:=S!(1, 2)(3, 6)(4, 8)(5, 9)(7, 11);
yy:=S!(1, 3, 11)(2, 7, 6)(4, 8, 12)(5, 9, 10);
N:=sub<S|xx,yy>;
f,N1,k:=CosetAction(N,sub<N|Id(N)>);
G<x,y,t>:=Group<x,y,t|x^2,y^3, (y^-1*x)^2,t^2,(t,x*y^-1),
(x*t)^7,(x * y * t^x)^2,(y*t)^9>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
M:=MaximalSubgroups(G1);
#M;
/*3*/
for i in [1..3] do #M[i]` subgroup; end for;
/*14
18*
56*/

/* only 18 is the candidates for maximal subgroup
divisible by N (#N = 6) we will perform DCE using this
 maximal subgroup. */

  %------------------------------------------------
```

```
C:=Conjugates(G1,M[2]`subgroup);
#C;
/*28*/

M:=MaximalSubgroups(G1);
#M;
/*3*/
M;
C:=Conjugates(G1,M[2]`subgroup);  /* 28 */
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i] then i;
end if; end for;
/*13*/

CC[28];

for g in G1 do if CC[13] eq sub<G1|f(x),f(y),g> then A:=g;
break;
end if; end for;
Order(A);
for g in G1 do if Order(g) eq 2 and
CC[13] eq sub<G1|f(x),f(y),g> then A:=g; break; end if;
end for;

 %----------------------------------------------

W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
/*function(W)
    w4 := W.3 * W.1; w5 := w4 * W.3; w6 := w5 *
    W.2; w7 := w6 * W.3; w8 := w7 *
    W.1; w9 := w8 * W.3; w10 := w9 * W.2; w11 := w10 *
    W.3; w12 := w11 * W.1;
    w13 := w12 * W.3; w14 := w13 * W.2; w15 := w14 *
    W.3; return w15;
end function*/

/* now we run the loop to change the permutation
 into words but shorter relation*/

N:=G1;
NN:=G;
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#G1]];
for i in [2..#G1] do
 P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
end for;
PP:=Id(N);
for k in [1..#P] do PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
/*t * x * t * y * t * x * t * y * t * x * t * y * t*/

/*H1=H we have the relation of H and the order of H is
18 which is what we have in M[2]*/
 H1:=sub<G1|f(x),f(y),f(t * x * t * y * t * x * t * y * t * x
 * t * y * t)>;
#H1;
/*18*/

 %-----------------------------------------------

/*now lets find out the composition factor of H=M[2]*/
CompositionFactors(M[2]`subgroup);
/*  G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
*/
NL:=NormalLattice(M[2]`subgroup);
NL;
/*Normal subgroup lattice
-----------------------

[4]  Order 18  Length 1  Maximal Subgroups: 3
---
[3]  Order 9   Length 1  Maximal Subgroups: 2
```

```
---
[2]  Order 3   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
*/
Center(M[2]`subgroup);
for i in [1..#NL] do if IsAbelian(NL[i]) then i;
end if; end for;
/*1
2
3
*/
NL[3];
/*The largest abelian group of M is NL[3] of order  9.
Now we factor by NL[3]*/
X:=[3^2];
IsIsomorphic(NL[3],AbelianGroup(GrpPerm,X));
/* isomorphism type of M is 3^2:2*/

 %-----------------------------------------------

/*/////Maximal Subgroup Clean File with my new
control group Sym_3/////////*/
S:=Sym(3);
xx:=S!(1, 2);
yy:=S!(1, 2, 3);
/*we are working with the new generators of my
new control group S_3*/
N:=sub<S|xx,yy>;
#N;
Set(N);
G<x,y,t>:=Group<x,y,t|x^2,y^3, (y^-1*x)^2,t^2,(t,x*y^-1),
(x*t)^7,(x * y * t^x)^2,(y*t)^9>;
#G;
 H:=sub<G|x,y,t * x * t * y * t * x
  * t * y * t * x * t * y * t>;
#H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
IM:=sub<G1|f(x),f(y),f(t * x * t * y * t * x *
 t * y * t * x * t * y * t)>;
#IN;#IM;
/*6, 18*/
```

```
 %-----------------------------------------------
 /*Now we construct a double coset enumeration*/

ts:=[Id(G1): i in [1..3]];
ts[1]:=f(t); ts[2]:=f(t^(x));
ts[3]:=f(t^(y^2));
#DoubleCosets(G,H,sub<G|x,y>);
/*7*/
DoubleCosets(G,H,sub<G|x,y>);
/*{ <GrpFP: H, Id(G), GrpFP>,
<GrpFP: H, t * x * t * x * t * y * t * y * t,
GrpFP>, <GrpFP: H, t * x * t * x * t, GrpFP>,
<GrpFP: H, t, GrpFP>, <GrpFP: H, t
* x * t, GrpFP>, <GrpFP: H, t * x * t * y * t, GrpFP>,
<GrpFP: H, t * x * t * x * t * y * t, GrpFP> }*/

/*Expand relations
1) (x*t)^7= x^7*t^x^6*t^x^5*t^x^4*t^x^3*t^x^2*t^xt=1
1=(1, 2)t_1*t_2*t_1*t_2t_1*t_2t_1
ts[1]*ts[2]*ts[1] eq f(x)*ts[1]*ts[2]*ts[1]*ts[2];

2) 1=(x * y * t^x)^2  xyt^xxyt^x= xy*xy*(xy)^-1t^x*xyt^x
=(x*y)^2t^x^2*y*t^x
=t_2t_2
=1

3)1=(y*t)^9
=y^9*t^y^8*t^y^7*t^y^6*t^y^5*t^y^4*t^y^3*t^y^2*t^y*t
1=y^9*t^y^8*t^y^7*t^y^6*t^y^5*t^y^4*t^y^3*t^y^2*t^y*t
t_3*t_2*t_1*t_3*t_2*t_1*t_3t_2t_1=1
ts[3]*ts[2]*ts[1]*ts[3] eq ts[1]*ts[2]*ts[3]*ts[1]*ts[2];
H=9:2
N=S_3
G1=PSL_2(8)
2^3:S_3 */

 %-----------------------------------------------

Index(G,H);
prodim := function(pt, Q, I)

 v := pt;
 for i in I do
   v := v^(Q[i]);
```

```
 end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]] where
null is [Integers() | ];
  for i := 1 to 3 do
   cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..28] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*3*/


N1:=Stabiliser (N,[1]);
SSS:={[1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do for n in IM do if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1; #N1; /* #N1=2 and N1=(2, 3)*/
T1:=Transversal(N,N1);
 for i in [1..#T1] do ss:=[1]^T1[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..28] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*3*/
Orbits(N1);
/* GSet{@ 1 @},
    GSet{@ 2, 3 @}*/


N12:=Stabiliser(N,[1,2]);
 SSS:={[1,2]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;
```

```
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N12; #N12;
T12:=Transversal(N,N12);
 for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..28] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*9*/
Orbits(N12);
/*GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @}*/

/**********************[1,2]*************************/
/*121 a new double coset, 122 one transversal goes back to
 [1], and 123 a new double coset*/
N121:=Stabiliser(N,[1,2,1]);
 SSS:={[1,2,1]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*
 ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N121; #N121;
T121:=Transversal(N,N121);
 for i in [1..#T121] do ss:=[1,2,1]^T121[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..28] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
/*15*/
```

```
Orbits(N121);
/*  GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @}*/

N123:=Stabiliser(N,[1,2,3]);
 SSS:={[1,2,3]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*
 ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N123; #N123;

T123:=Transversal(N,N123);
 for i in [1..#T123] do ss:=[1,2,3]^T123[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..28] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*21*/
Orbits(N123);
/* GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @}
*/

/*****************[1,2,1]*************************/
/*1212 lives  on a different double coset and 1211
 lives in [12]*/
N1213:=Stabiliser(N,[1,2,1,3]);
 SSS:={[1,2,1,3]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;
```

```
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]
 *ts[Rep(Seqq[i])[2]]
 *ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N1213; #N1213;
/*[ 1, 2, 1, 3 ]
[ 3, 2, 3, 1 ] #N1213=2*/

/* Equal Name */
N1213s:=N1213;
for n in N do if 1^n eq 3 and 2^n eq 2 and 1^n eq
 3 and 3^n eq 1  then
N1213s:=sub<N|N1213s,n>;
 end if; end for;
 #N1213s;
N1213s;
/*(1, 3)*/
[1,2,1,3]^N1213s;

N1213:=Stabiliser(N,[1,2,1,3]);
N1213;
 N1213:=sub<N|(1,3)>;
#N1213;
[1,2,1,3]^N1213;

T:=Transversal(N,N1213);
for i in [1..#T] do {[1,2,1,3]^N1213}^T[i];
 end for;
for n in IM do if ts[1]*ts[2]*ts[1]*ts[3] eq
n*ts[3]*ts[2]*ts[3]*ts[1] then n; end if; end for;

ts[1]*ts[2]*ts[1]*ts[3] eq
f(x * y^-1 * t * x * t * y * t * x * t * y * t^-1 * y^-1
* t^-1 * x^-1 * t^-1)*ts[3]*ts[2]*ts[3]*ts[1];

/* Add Relation */
for n in IM do if ts[1]*ts[2]*ts[1]*ts[3]
eq n*ts[3]*ts[2]*ts[3]*ts[1] then n; end if; end for;
for n in IM do if ts[1]*ts[2]*ts[1]*ts[3] eq
```

```
n*ts[3]*ts[2]*ts[3]*ts[1] then A:=n; end if; end for;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
g:=function(W);
    return W.1;
end function;
g(G);
/*a*/
T1213:=Transversal(N,N1213);
 for i in [1..#T1213] do ss:=[1,2,1,3]^T1213[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..28] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*24*/
Orbits(N1213);
/*GSet{@ 2 @},
    GSet{@ 1, 3 @}
*/
```

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